Systems Laboratory, Spring 2025

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1D convolution in continuous time

Contents map

developed content units	taxonomy levels
convolution	u1, e1

prerequisite content units	taxonomy levels
signal	u1, e1

Main ILO of sub-module "1D convolution in continuous time"

Compute the convolution between two single dimensional continuous time signals

Why convolution?

because for a LTI system with impulse response $h(t)$
it follows that $y_{\text{forced}}(t) = u * h(t)$

extremely important result for LTI systems:

$$y_{\text{forced}}(t) = h * u(t) = u * h(t) :=$$

$$:= \int_{-\infty}^{+\infty} u(\tau) h(t-\tau) d\tau = \int_{-\infty}^{+\infty} h(\tau) u(t-\tau) d\tau$$

. . . and this module = what that formula actually means from graphical perspectives

Additional material

Videos:

- https://www.youtube.com/watch?v=KuXjwB4LzSA
- https://www.youtube.com/watch?v=acAw5WGtzuk
- https://www.youtube.com/watch?v=IaSGqQa5O-M (for connections with probability)
- https://www.youtube.com/playlist?list=PL4iThgVpN7hmbIhHnCa7SDOOgLMoNwED_
- https://www.youtube.com/playlist?list=PL4mJLdGEHNvhCuPXsKFrnD7AaQBlMEB6a

Animations:

- https://lpsa.swarthmore.edu/Convolution/CI.html
- https://phiresky.github.io/convolution-demo/

Towards decomposing this formula in pieces

$$y_{\text{forced}}(t) = h * u(t) = \int_{-\infty}^{+\infty} u(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

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better focus on

$$\int_{-\infty}^{+\infty} u(\tau)h(t-\tau)d\tau$$

or on

$$\int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau \text{ ?}$$

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in automatic control typically better the second

Towards decomposing this formula in pieces, small change of notation

$$y_{\text{forced}}(t) = \int_{-\infty}^{+\infty} h(\tau) u(t - \tau) d\tau \quad \mapsto \quad y_{\text{forced}}(\text{now}) = \int_{-\infty}^{+\infty} h(\tau) u(\text{now} - \tau) d\tau$$

Decomposing this formula in pieces

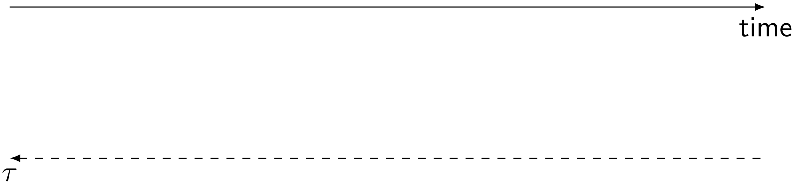
$$y_{\text{forced}}(\text{now}) = \int_{-\infty}^{+\infty} h(\tau)u(\text{now} - \tau)d\tau$$

$\implies$ constituent pieces =

- $u(\text{now} - \tau)$
- $h(\tau)$
- $u(\text{now} - \tau)h(\tau)$
- $\int u(\text{now} - \tau)h(\tau)d\tau$

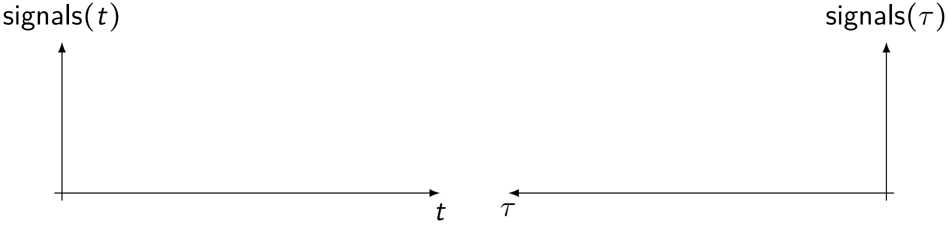
Visualizing the various pieces

$$u(\mathrm{now} - \tau) \qquad h(\tau) \qquad u(\mathrm{now} - \tau)h(\tau) \qquad \int_{-\infty}^{+\infty} h(\tau)u(\mathrm{now} - \tau)d\tau$$



Example

$$u(t) = \begin{cases} 1 & \text{for } t \in [1,2] \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 2 & \text{for } t \in [0,1) \\ 1 & \text{for } t \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$



Another Example:

$$u(t) = \begin{cases} 1 & \text{for } t \in [1,2] \text{ and } t \in [2,3] \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 2 & \text{for } t \in [0,2) \\ 0 & \text{otherwise} \end{cases} \qquad ?$$

Paramount message

h in $y_{\text{forced}}(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$ represents how much the past u's contribute to the current y_{forced}:

Refreshing what we are doing and why

$$\text{Dynamics of a cart: } \dot{v}(t) = -\frac{k}{m}v(t) + \frac{k}{m}F(t) \text{ with:}$$

- **control input:** $u(t)$ (actuation from the motor, in this case $= F(t)$)

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- **forced response:** $y_{\text{forced}}(t) = u * h(t)$ (output in time corresponding to null initial condition, i.e., $y(0) = 0$, and input $u(t)$ whatever it is)

Refreshing what we are doing and why

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- **free evolution:** $y_{\mathrm{free}}(t)$ (output in time corresponding to no input, i.e., $u(t) = 0$, and initial condition $y(0)$ whatever it is)
- **forced response:** $y_{\mathrm{forced}}(t) = u * h(t)$ (output in time corresponding to null initial condition, i.e., $y(0) = 0$, and input $u(t)$ whatever it is)
- **total response:** $y(t) = y_{\mathrm{free}}(t) + y_{\mathrm{forced}}(t)$

Quiz time!

$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau) u(t - \tau) d\tau$$

- is $h * u(t) = u * h(t)$?

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- is $h * u(t) = u * h(t)$?
- is $(\alpha h_1 + \beta h_2) * u(t) = \alpha \left(h_1 * u(t)\right) + \beta \left(h_2 * u(t)\right)$?

Quiz time!

$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

- is $h * u(t) = u * h(t)$?
- is $(\alpha h_1 + \beta h_2) * u(t) = \alpha\,(h_1 * u(t)) + \beta\,(h_2 * u(t))$?
- if both $h(\tau) = 0$ and $u(t) = 0$ if $t < 0$, how can we simplify $y(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$?

Summarizing

Compute the convolution between two single dimensional continuous time signals

- take one of the two signals
- translate it to the “current t”
- flip it
- multiply the two signals in a pointwise fashion
- compute the integral of the result

Most important python code for this sub-module

Methods implementing (discrete) convolutions

- `https://numpy.org/doc/2.1/reference/generated/numpy.convolve.html`
- `https://docs.scipy.org/doc/scipy/reference/generated/scipy.signal.convolve.html`

Self-assessment material

Question 1

What does the convolution integral $y_{\text{forced}}(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$ represent in the context of LTI systems?

Potential answers:

I: The free evolution of the system output.
II: The forced response of the system output due to the input $u(t)$.
III: The total response of the system, including initial conditions.
IV: The impulse response of the system.
V: I do not know.

Question 2

Which of the following is true about the convolution operation $h * u(t)$?

Potential answers:

I: It is only defined for periodic signals.

II: It is only applicable to discrete-time systems.

III: It is commutative, i.e., $h * u(t) = u * h(t)$.

IV: It requires both signals to be symmetric.

V: I do not know.

Question 3

What does the impulse response $h(t)$ of an LTI system represent?

Potential answers:

I: The input signal $u(t)$ applied to the system.
II: The free evolution of the system output.
III: The total response of the system, including initial conditions.
IV: The output of the system when the input is a Dirac delta function $\delta(t)$.
V: I do not know.

Question 4

If $h(\tau) = 0$ for $\tau < 0$ and $u(t) = 0$ for $t < 0$, how can the convolution integral $y(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$ be simplified?

Potential answers:

I: $y(t) = \int_{0}^{t} h(\tau)u(t-\tau)d\tau$

II: $y(t) = \int_{0}^{+\infty} h(\tau)u(t-\tau)d\tau$

III: $y(t) = \int_{-\infty}^{+\infty} h(\tau)u(\tau)d\tau$

IV: $y(t) = \int_{-\infty}^{0} h(\tau)u(t-\tau)d\tau$

V: I do not know.

Question 5

What is the graphical interpretation of $h(\tau)$ in the convolution integral $y_{\text{forced}}(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$?

Potential answers:

I: It represents the future inputs of the system.
II: It represents how much past inputs contribute to the current output.
III: It represents the free evolution of the system.
IV: It represents the total energy of the system.
V: I do not know.

Recap of sub-module "1D convolution in continuous time"

- convolution is an essential operator, since it can be used for LTI systems to compute forced responses
- its graphical interpretation aids interpreting impulse responses as how the past inputs contribute to current outputs

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