Systems Laboratory, Spring 2025

Damiano Varagnolo –

what is the superposition principle, and what does it imply

Contents map

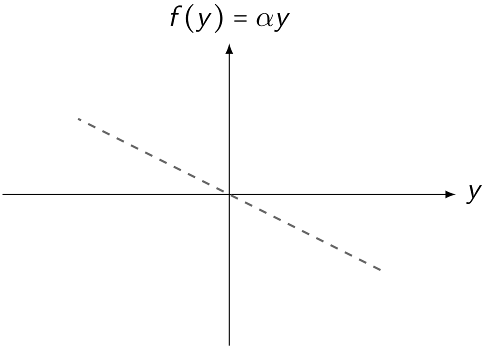
developed content units	taxonomy levels
superposition principle	u1, e1

prerequisite content units	taxonomy levels
LTI ODE	u1, e1

Main ILO of sub-module
"what is the superposition principle, and what does it imply"

Describe the importance of the super-
position principle to analyze LTI systems

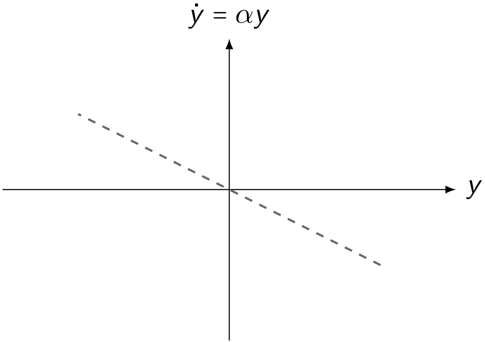
Starting with graphs



implications/definition of linearity:

- $f(x + y) = f(x) + f(y)$
- $f(\alpha y) = \alpha f(y)$

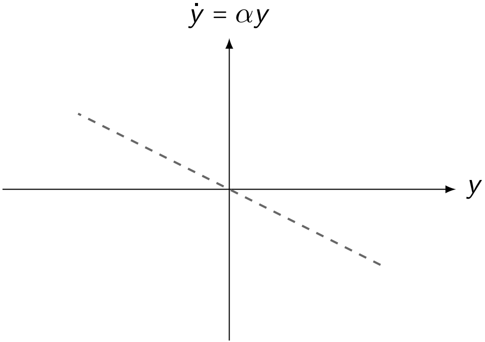
What if we interpret this as an ODE?



$\Longrightarrow$ an LTI system, for which

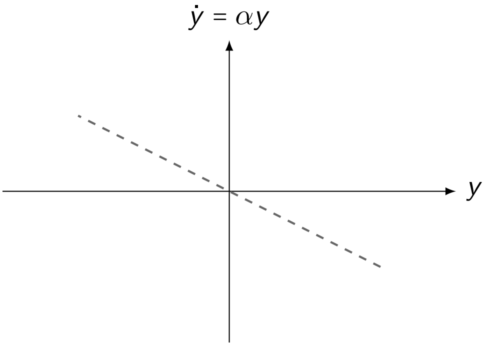
$$\dot{y} = \alpha y \quad \text{is solved by} \quad y(t) = y(0)\exp(\alpha t) \quad \forall y(0), \alpha, t$$

And can we build on top of this?



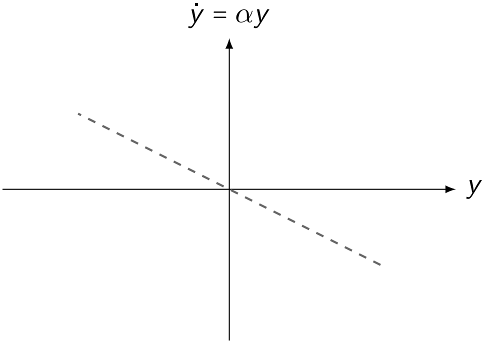
- $y'(0) = 2 \mapsto y'(t) = 2\exp(\alpha t)$

And can we build on top of this?



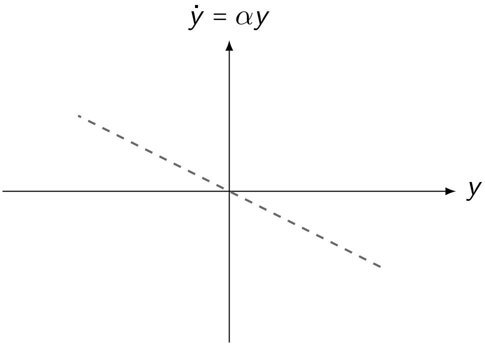
- $y'(0) = 2 \mapsto y'(t) = 2\exp(\alpha t)$
- $y''(0) = 3 \mapsto y''(t) = 3\exp(\alpha t)$

And can we build on top of this?



- $y'(0) = 2 \mapsto y'(t) = 2\exp(\alpha t)$
- $y''(0) = 3 \mapsto y''(t) = 3\exp(\alpha t)$
- $y'''(0) = 3 + 2 \mapsto y'''(t) = (3 + 2)\exp(\alpha t)$

And can we build on top of this?



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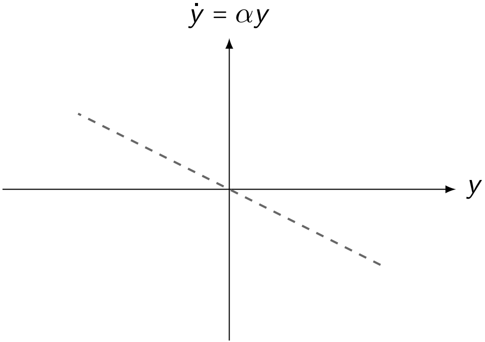

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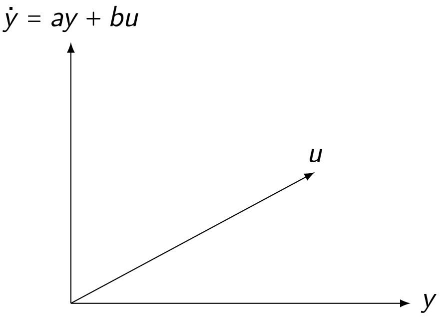
$$\boxed{y'(0) + y''(0) \mapsto y'(t) + y''(t)}$$

Further generalization



- $\left\{y'(0), u'\right\} \mapsto y'(t)$
- $\left\{y''(0), u''\right\} \mapsto y''(t)$
- $\left\{y'(0) + y''(0), u' + u''\right\} \mapsto y'(t) + y''(t)$

Aiding intuitions with math

Linearity implies that if $\{y', u', y'(0)\}$ and $\{y'', u'', y''(0)\}$ satisfy

$$\begin{cases} \dfrac{dy'(t)}{dt} & = ay'(t) + bu'(t) \\ y'(0) & = y'_0 \\ \dfrac{dy''(t)}{dt} & = ay''(t) + bu''(t) \\ y''(0) & = y''_0 \end{cases} \tag{1}$$

then their sum also satisfies

$$\begin{cases} \dfrac{d\big(\alpha' y'(t) + \alpha'' y''(t)\big)}{dt} & = a\big(\alpha' y'(t) + \alpha'' y''(t)\big) + b\big(\alpha' u'(t) + \alpha'' u''(t)\big) \\ \alpha' y'(0) + \alpha'' y''(0) & = \alpha' y'_0 + \alpha'' y''_0 \end{cases} \tag{2}$$

Rephrasing

Linearity implies that if $\{y', u', y'(0)\}$ and $\{y'', u'', y''(0)\}$ satisfy the ODE then also their sum $\{y' + y'', u' + u'', y'(0) + y''(0)\}$ satisfies the ODE.

Rephrasing

Linearity implies that if $\{y', u', y'(0)\}$ and $\{y'', u'', y''(0)\}$ satisfy the ODE then also their sum $\{y' + y'', u' + u'', y'(0) + y''(0)\}$ satisfies the ODE.

The superposition principle in words

in LTI systems
combining inputs and initial conditions
produces a total effect
that is the linear combination
of that effects
one would get with the individual causes
each acting separately

Important: the superposition principle works with any LTI

Will be repeated and stated again precisely later on

the proof holds for every system that generalizes $\dot{y} = ay + bu$,
i.e., every "linear combination of dots of y = linear combination of dots of u"

Superposition principle $\implies$ response of LTIs $=$ free evolution $+$ forced response

Superposition principle $\Longrightarrow$ response of LTIs = free evolution + forced response

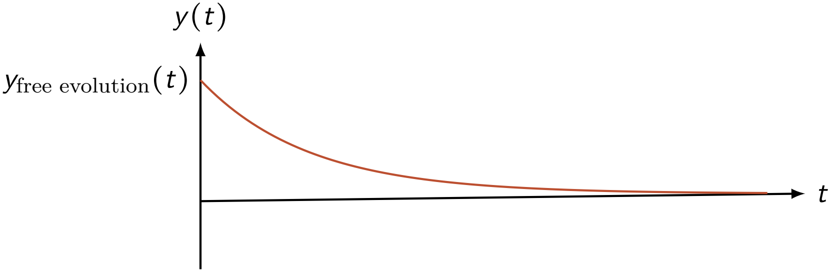
assume:

- $\dot{y} = ay + bu$
-
-

Superposition principle $\Longrightarrow$ response of LTIs $=$ free evolution $+$ forced response

assume:

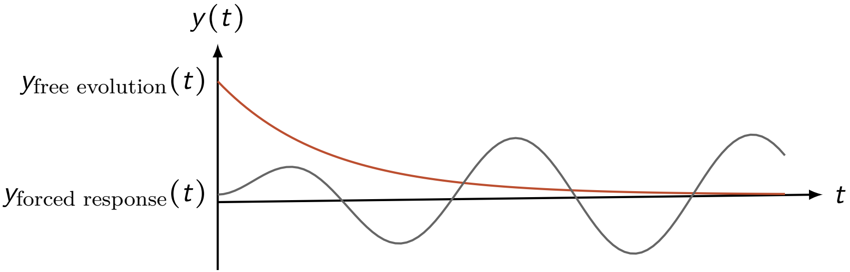
- $\dot{y} = ay + bu$
- $\{u(t) = 0(t), \quad y(0) \neq 0\}$ causes $y_{\text{free evolution}}(t)$
-



Superposition principle $\implies$ response of LTIs = free evolution + forced response

assume:

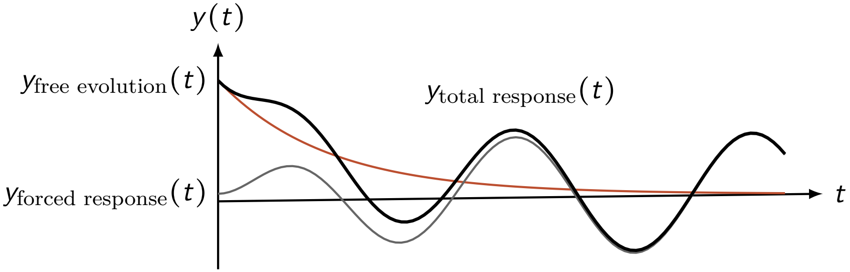
- $\dot{y} = ay + bu$
- $\{u(t) = 0(t), \quad y(0) \neq 0\}$ causes $y_{\text{free evolution}}(t)$
- $\{u(t) \neq 0(t), \quad y(0) = 0\}$ causes $y_{\text{forced response}}(t)$



Superposition principle $\Longrightarrow$ response of LTIs = free evolution + forced response

assume:

- $\dot{y} = ay + bu$
- $\{u(t) = 0(t), \quad y(0) \neq 0\}$ causes $y_{\text{free evolution}}(t)$
- $\{u(t) \neq 0(t), \quad y(0) = 0\}$ causes $y_{\text{forced response}}(t)$



then $\{u(t) \neq 0(t), y(0) \neq 0\}$ causes $y_{\text{free evolution}}(t) + y_{\text{forced response}}(t)$

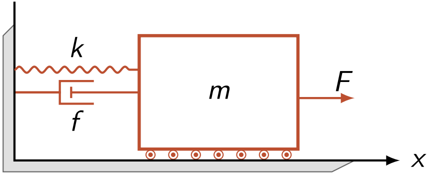
A mnemonic scheme

(only for LTI systems!!)

$$(u, y_0) = (0, y_0) + (u, 0)$$

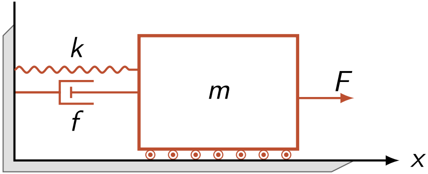
total response = free evolution + forced response

Continuing with some intuitions



Discussion: how will the cart move if I use $u(t) = \sin(\omega t)$ starting from a resting state? (only intuitively, assuming everything ideal)

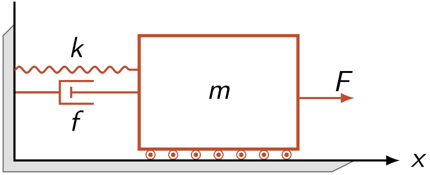
Continuing with some intuitions



Discussion: how will the cart move if I use $u(t) = \sin(\omega t)$ starting from a resting state? (only intuitively, assuming everything ideal)

And what about if $u(t) = 2\sin(\omega t)$?

Continuing with some intuitions



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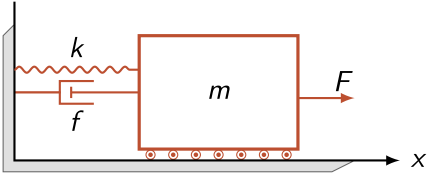
Continuing with some intuitions



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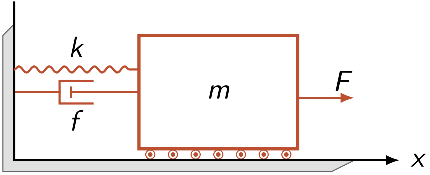
And what about $u(t) = \alpha' \sin\left(\omega' t\right) + \alpha'' \sin\left(\omega'' t\right)$?

Continuing with some intuitions



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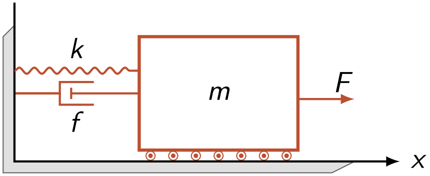
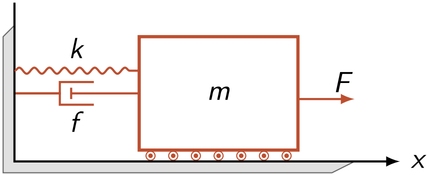
Refining the intuitions



Assume to have measured

$$\boldsymbol{y}'(0), u'(t) \mapsto y'(t) \qquad \boldsymbol{y}''(0), u''(t) \mapsto y''(t)$$

Refining the intuitions



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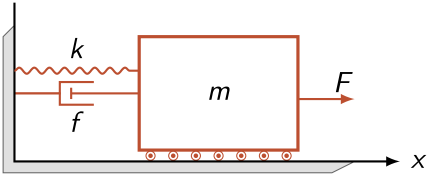
$$\boldsymbol{y}'(0), u'(t) \mapsto y'(t) \qquad \boldsymbol{y}''(0), u''(t) \mapsto y''(t)$$

Saying "this system is linear" means assuming $\forall \alpha', \alpha'' \in \mathbb{R}$

$$\alpha' \boldsymbol{y}'(0) \alpha' \boldsymbol{y}''(0), u'(t) \mapsto y'(t) \qquad \boldsymbol{y}''(0), u''(t) \mapsto y''(t)$$

Refining the intuitions



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Saying "this system is linear" means assuming $\forall \alpha', \alpha'' \in \mathbb{R}$

$$\alpha' \boldsymbol{y}'(0) \alpha' \boldsymbol{y}''(0), u'(t) \mapsto y'(t) \qquad \boldsymbol{y}''(0), u''(t) \mapsto y''(t)$$

thus assuming that from a resting state the input $u(t) = \alpha \sin(\omega_\alpha t) + \beta \sin(\omega_\beta t)$ causes $y(t) = \alpha y_{\omega\alpha}(t) + \beta y_{\omega\beta}(t)$

Summarizing

Describe the importance of the super-
position principle to analyze LTI systems

- it makes us able to say "total $=$ free $+$ forced"

Most important python code for this sub-module

Suggestion

part of the SciPy library (`scipy.signal`) provides tools for working with LTI systems, including creating transfer functions, state-space representations, and analyzing system responses (stuff that will be seen in the next modules)

Self-assessment material

Question 1

What does the superposition principle imply for LTI systems?

Potential answers:

I: The total response is the product of the free evolution and forced response.
II: The total response is the sum of the free evolution and forced response.
III: The total response is independent of the initial conditions.
IV: The total response is only determined by the input.
V: I do not know.

Question 2

Which of the following is a necessary condition for the superposition principle to hold in a system?

Potential answers:

I: The system must be nonlinear.
II: The system must be linear and time-invariant.
III: The system must have time-varying parameters.
IV: The system must be unstable.
V: I do not know.

Question 3

What is the free evolution of an LTI system?

Potential answers:

I: The response of the system to a nonzero input with zero initial conditions.
II: The response of the system to zero input with nonzero initial conditions.
III: The steady-state response of the system.
IV: The transient response of the system.
V: I do not know.

Question 4

If an LTI system has an input $u(t) = \alpha' u'(t) + \alpha'' u''(t)$ and initial conditions $y(0) = \alpha' y'(0) + \alpha'' y''(0)$, what is the total response $y(t)$?

Potential answers:

I: $y(t) = \alpha' y'(t) \cdot \alpha'' y''(t)$
II: $y(t) = \alpha' y'(t) + \alpha'' y''(t)$
III: $y(t) = \alpha' y'(t) - \alpha'' y''(t)$
IV: $y(t) = \alpha' y'(t) / \alpha'' y''(t)$
V: I do not know.

Question 5

What is the forced response of an LTI system?

Potential answers:

I: The response of the system to a nonzero input with zero initial conditions.
II: The response of the system to zero input with nonzero initial conditions.
III: The response of the system to a step input.
IV: The response of the system to a sinusoidal input.
V: I do not know.

Recap of sub-module “what is the superposition principle, and what does it imply”

- superposition principle helps logically separating specific causes into specific effects
- linear ODEs $\Longrightarrow$ superposition principle
- superposition principle $\Longrightarrow$ "whole $=$ free $+$ forced"
- nonlinear systems WON'T satisfy this principle!

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