Systems Laboratory, Spring 2025

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how to linearize an ODE

Contents map

developed content units	taxonomy levels
linearization	u1, e1

prerequisite content units	taxonomy levels
ODE	u1, e1

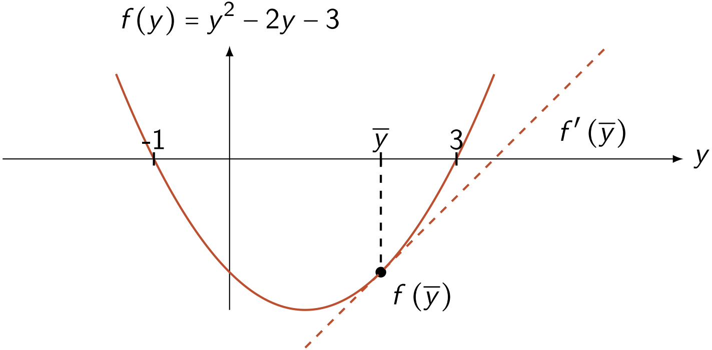
Main ILO of sub-module "how to linearize an ODE"

Linearize a nonlinear ODE around an equilibrium point

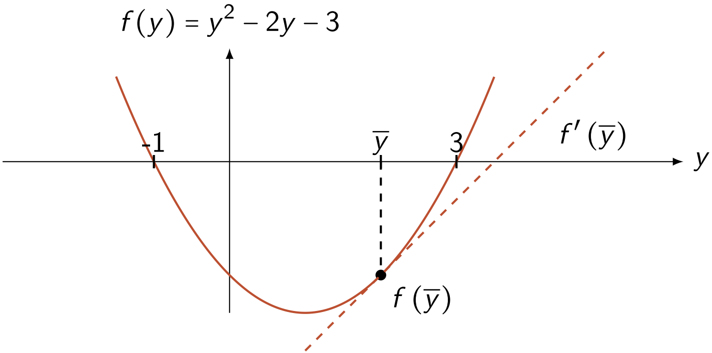
The path towards linearizing a model

- what does linearizing a function mean?
- what does linearizing a model mean?
- how shall we linearize a model?

What does linearizing a scalar function mean?



What does linearizing a scalar function mean?



$$f(y) \approx f(\overline{y}) + \left.\frac{\partial f}{\partial y}\right|_{\overline{y}} (y - \overline{y})$$

(but the approximation is valid only close to the linearization point)

Obvious requirement

(but sometimes people forget about it . . .)

to compute the approximation

$$f(y) \approx f(\overline{y}) + \left.\frac{\partial f}{\partial y}\right|_{\overline{y}} (y - \overline{y})$$

the derivative of f at $\overline{y}$ must be defined. *(notation: $f \in C^n$ means that f has all its derivatives up to order n defined in $\mathbb{R}$. $f \in C^n(\mathcal{X})$ means defined in $\mathcal{X} \subseteq \mathbb{R}$)*

What does linearizing a vectorial function mean?

$$\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^m, \quad \boldsymbol{f} \in C^1 \quad \text{enables computing} \quad \boldsymbol{f}(\boldsymbol{y}) \approx \boldsymbol{f}(\boldsymbol{y}_0) + \nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0} (\boldsymbol{y} - \boldsymbol{y}_0)$$

linearize $\Longrightarrow$ *approximate each component!*

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linearize $\Longrightarrow$ *approximate each component!*

Discussion: then $\nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0}$ must be a matrix. Of which dimensions?

Example: linearize $\boldsymbol{f}$ around $\boldsymbol{y}_0$

$$\boldsymbol{f}\left(\boldsymbol{y}(t)\right) = \begin{bmatrix} \sin\left(y_1(t)\right) + \cos\left(y_2(t)\right) \\ \exp\left(y_1(t)y_2(t)\right) \end{bmatrix} \qquad \boldsymbol{y}_0 = \boldsymbol{y}(0) = [0, \pi]$$

And what if the vectorial function depends on more than one variable?

Assuming $\boldsymbol{f}$ differentiable in $\boldsymbol{y}_0, \boldsymbol{u}_0$,

$$\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) \approx \boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{u}_0) + \nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0} (\boldsymbol{y} - \boldsymbol{y}_0) + \nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0} (\boldsymbol{u} - \boldsymbol{u}_0)$$

with both $\nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0}$ and $\nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0}$ matrices of opportune size.

And what if the vectorial function depends on more than one variable?

Assuming $\boldsymbol{f}$ differentiable in $\boldsymbol{y}_0, \boldsymbol{u}_0$,

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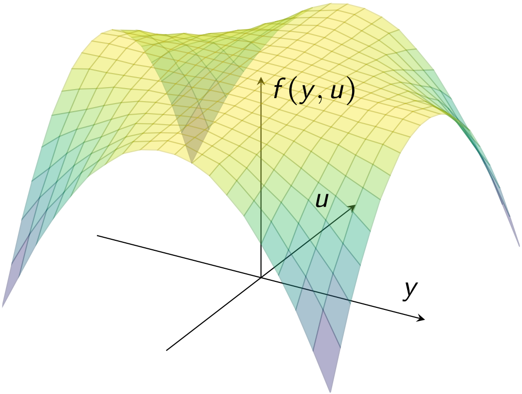
with both $\nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0}$ and $\nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0}$ matrices of opportune size. Alternative notation:

$$\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) \approx \boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{u}_0) + \nabla \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})\Big|_{\boldsymbol{y}_0, \boldsymbol{u}_0} \begin{bmatrix} \Delta \boldsymbol{y} \\ \Delta \boldsymbol{u} \end{bmatrix}$$

Graphical example with a $\mathbb{R}^2 \mapsto \mathbb{R}$ function

$$f(y,u) \approx f(y_0,u_0) + \nabla f(y,u)\Big|_{y_0,u_0} \begin{bmatrix} \Delta x \\ \Delta u \end{bmatrix}$$



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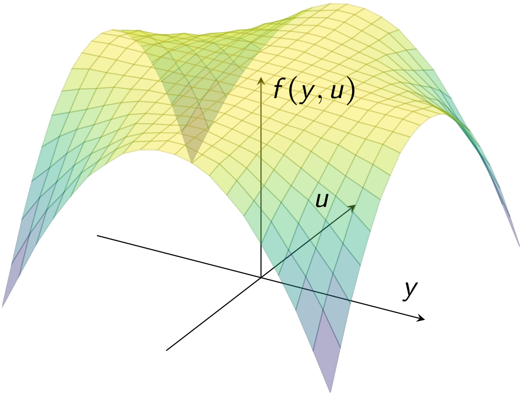

if $\boldsymbol{f} = [f_1, f_2]$ *then have two distinct plots, but the concept is the same*

Thus, linearization = stopping the Taylor series at order one

$$f \in C^M(\mathbb{R}) \quad \Longrightarrow \quad f(y) \approx \sum_{m=0}^{M} \frac{f^{(m)}(y_0)}{m!}(y - y_0)^m$$

Thus, linearization = stopping the Taylor series at order one

$$f \in C^M(\mathbb{R}) \qquad \Longrightarrow \qquad f(y) \approx \sum_{m=0}^{M} \frac{f^{(m)}(y_0)}{m!}(y - y_0)^m$$

multivariable extension = less neat formulas, but the concept is the same. The most

important case for our purposes:

$$\boldsymbol{f} \in C^1(\mathbb{R}^n, \mathbb{R}^m) \qquad \Longrightarrow \qquad \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) \approx \boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{u}_0) + \nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0}(\boldsymbol{y} - \boldsymbol{y}_0) + \nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{u}_0}(\boldsymbol{u} - \boldsymbol{u}_0)$$

?

What does linearizing an ODE mean?

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) \quad \approx \quad \dot{\widetilde{\boldsymbol{y}}} = A\widetilde{\boldsymbol{y}} + B\widetilde{\boldsymbol{u}}$$

linearize $\Longrightarrow$ *approximate the dynamics!*

Discussion: what is the simplest way to make this linear?

$$\dot{y} = ay + bu^{2/3}$$

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$$\dot{y} = ay + bu^{2/3}$$

Another discussion: can we apply the same "linearization trick" to $\dot{y} = a\sqrt{y} + bu$?

Discussion: why do we linearize nonlinear systems?

Discussion: where do we linearize nonlinear systems?

Linearization procedure - continuous time systems

$$\left(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) \text{ equilibrium} \quad \Longrightarrow \boldsymbol{f}\left(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) = 0$$

Linearization procedure - continuous time systems

$$\left(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) \text{ equilibrium} \implies \boldsymbol{f}\left(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) = 0$$

Procedure (assuming that the Taylor expansion exists):

- consider $\boldsymbol{y} = \boldsymbol{y}_{\text{eq}} + \Delta\boldsymbol{y}$, and $\boldsymbol{u} = \boldsymbol{u}_{\text{eq}} + \Delta\boldsymbol{u}$
- compute

$$\boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right) \approx \boldsymbol{f}\left(\boldsymbol{y}_0, \boldsymbol{u}_0\right) + \nabla_{\boldsymbol{y}}\boldsymbol{f}\big|_{\boldsymbol{y}_0}\left(\boldsymbol{y} - \boldsymbol{y}_0\right) + \nabla_{\boldsymbol{u}}\boldsymbol{f}\big|_{\boldsymbol{u}_0}\left(\boldsymbol{u} - \boldsymbol{u}_0\right)$$

setting though $\boldsymbol{y}_0 = \boldsymbol{y}_{\text{eq}}$

Linearization procedure - continuous time systems

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$$\boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right) \approx \boldsymbol{f}\left(\boldsymbol{y}_0, \boldsymbol{u}_0\right) + \nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0} \left(\boldsymbol{y} - \boldsymbol{y}_0\right) + \nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{u}_0} \left(\boldsymbol{u} - \boldsymbol{u}_0\right)$$

setting though $\boldsymbol{y}_0 = \boldsymbol{y}_{\text{eq}}$

$$\Longrightarrow \qquad \frac{\partial\left(\boldsymbol{y}_{\text{eq}} + \Delta \boldsymbol{y}\right)}{\partial t} \approx \boldsymbol{f}\left(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) + \nabla \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right)\Big|_{\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \begin{bmatrix} \Delta \boldsymbol{y} \\ \Delta \boldsymbol{u} \end{bmatrix}$$

Linearization procedure - continuous time systems

$$(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) \text{ equilibrium} \implies \boldsymbol{f}(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) = 0$$

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setting though $\boldsymbol{y}_0 = \boldsymbol{y}_{\text{eq}}$

$$\implies \qquad \frac{\partial (\boldsymbol{y}_{\text{eq}} + \Delta \boldsymbol{y})}{\partial t} \approx \boldsymbol{f}(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) + \nabla \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})\Big|_{\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \begin{bmatrix} \Delta \boldsymbol{y} \\ \Delta \boldsymbol{u} \end{bmatrix}$$

note then that $\dfrac{\partial (\boldsymbol{y}_{\text{eq}} + \Delta \boldsymbol{y})}{\partial t} = \Delta \dot{\boldsymbol{y}}$ and that $\boldsymbol{f}(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) = \boldsymbol{0}$

Linearization procedure - continuous time systems

$$(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) \text{ equilibrium} \quad \Longrightarrow$$

$$\Delta\dot{\boldsymbol{y}} \approx \nabla_{\boldsymbol{y}} \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})\Big|_{\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta\boldsymbol{y} + \nabla_{\boldsymbol{u}} \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})\Big|_{\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta\boldsymbol{u}$$

and, since

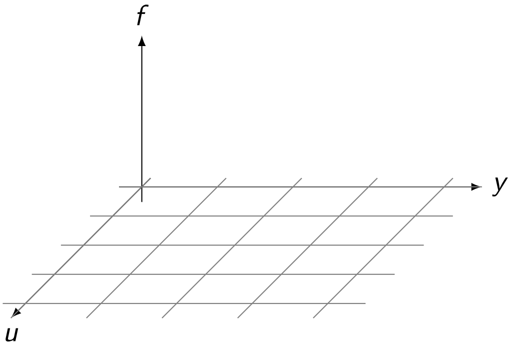
- the two ∇'s are matrices, and
- this is an approximate dynamics,

it follows that the approximated system is

$$\Delta\dot{\widetilde{\boldsymbol{y}}} = A\Delta\widetilde{\boldsymbol{y}} + B\Delta\boldsymbol{u}$$

What does this mean graphically?

$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right) \qquad \text{vs.} \qquad \Delta\dot{\widetilde{\boldsymbol{y}}} = A\Delta\widetilde{\boldsymbol{y}} + B\Delta\widetilde{\boldsymbol{u}}$$



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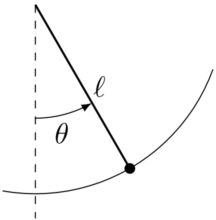
A from-start-to-end example: the pendulum

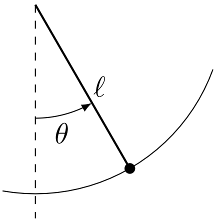


First step: equations of motion:

- gravity: $F_g = -mg\sin(\theta)$
- friction: $F_f = -f\ell\dot{\theta}$
- input torque: $F_u = u/\ell$

resulting dynamics: $m\ell\ddot{\theta} = -mg\sin(\theta) - f\ell\dot{\theta} + \frac{u}{\ell}$

First step: transforming this in a state space system



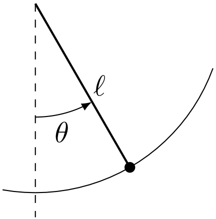
thus from

$$m\ell\ddot{\theta} = -mg\sin(\theta) - f\ell\dot{\theta} + \frac{u}{\ell}$$

into

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -\frac{g}{\ell}\sin\left(y_1\right) - \frac{f}{m}y_2 + \frac{1}{m\ell^2}u$$

Second step: finding the equilibria (assuming $u = 0$)



thus from

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -\frac{g}{\ell}\sin(y_1) - \frac{f}{m}y_2 + \frac{1}{m\ell^2}u \end{aligned}$$

to

$$\begin{cases} 0 = y_2 \\ 0 = -\frac{g}{\ell}\sin(y_1) - \frac{f}{m}y_2 \end{cases} \implies \quad \boldsymbol{y}_{\text{eq.inst}} = \begin{bmatrix} \pi + 2k\pi \\ 0 \end{bmatrix}, \qquad \boldsymbol{y}_{\text{eq.st}} = \begin{bmatrix} 0 + 2k\pi \\ 0 \end{bmatrix}$$

Linearizing around the first equilibrium

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -\frac{g}{\ell}\sin(y_1) - \frac{f}{m}y_2 + \frac{1}{m\ell^2}u$$

linearizing around $\boldsymbol{y}_{\text{eq.st}} = [0, 0]^T$, $u = 0$ implies

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}\right|_{\boldsymbol{y}_{\text{eq}\alpha}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{f}{m} \end{bmatrix}$$

$$B = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\right|_{\boldsymbol{y}_{\text{eq}\alpha}} = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}$$

Linearizing around the second equilibrium

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -\frac{g}{\ell}\sin(y_1) - \frac{f}{m}y_2 + \frac{1}{m\ell^2}u$$

linearizing around $\boldsymbol{y}_{\text{eq}\beta} = [\pi, 0]^T$, $u = 0$ implies

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}\right|_{\boldsymbol{y}_{\text{eq}\beta}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{f}{m} \end{bmatrix}$$

$$B = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\right|_{\boldsymbol{y}_{\text{eq}\beta}} = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}$$

The two linearized systems

Around the stable equilibrium: $$\begin{bmatrix} \Delta \dot{y}_1 \\ \Delta \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{f}{m} \end{bmatrix} \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} u$$

Around the unstable equilibrium: $$\begin{bmatrix} \Delta \dot{y}_1 \\ \Delta \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{f}{m} \end{bmatrix} \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} u$$

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the trajectories starting close to the stable equilibrium "stay around there", while the trajectories starting close to the unstable equilibrium "run away". This is because of the inner structure of the two state update matrices – another reason why we shall study linear algebra

?

Summarizing the procedure

- linearizing $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})$ is meaningful only around an equilibrium $(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$

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Summarizing the procedure

- linearizing $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})$ is meaningful only around an equilibrium $(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$
- to find the equilibria of a system we need to solve $\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) = \boldsymbol{0}$
- each equilibrium will lead to its "own" corresponding linear model $\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$, where A and B thus depend on $(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$ and $\boldsymbol{y}$, $\boldsymbol{u}$ in $\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$ have actually the meaning of $\Delta\boldsymbol{y}$, $\Delta\boldsymbol{u}$ with respect to the equilibrium

Summarizing the procedure

- linearizing $\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right)$ is meaningful only around an equilibrium $\left(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right)$
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- each linearized model $\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$ is more or less valid only in a neighborhood of $\left(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right)$. Moreover the size of this neighborhood depends on the curvature of $\boldsymbol{f}$ around that specific equilibrium point

Recapping the rationale behind linearization

- linear systems are easier to analyze than nonlinear systems

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- modal analysis and rational Laplace-transforms call for linear systems

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- linear systems are easier to analyze than nonlinear systems
- modal analysis and rational Laplace-transforms call for linear systems
- many advanced control techniques are based on linear systems

linearization = a very useful tool to do
analysis and design of control systems

Linearization - Another example

electrostatic microphone:

- q = capacitor charge
- h = distance of armature from its natural equilibrium
- $\boldsymbol{y} = \left[q, h, \dot{h}\right]$
- R = circuit resistance
- E = voltage generated by the generator (constant)
- C = capacity of the capacitor
- m = mass of the diaphragm + moved air
- k = mechanical spring coefficient
- β = mechanical dumping coefficient
- u_1 = incoming acoustic signal

Linearization - Another example

a physics-driven model:

$$\begin{cases} \dot{y}_1 &= -\dfrac{1}{Ra} y_1 \left(L + y_2 \right) + \dfrac{E}{R} \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= -\dfrac{\beta}{m} y_3 - \dfrac{k}{m} y_2 - \dfrac{y_1^2}{2am} + \dfrac{1}{m} u_1 \end{cases}$$

Linearization - Example

1-st step: compute the equilibria

$$\begin{cases} \dot{y}_1 &= -\frac{1}{Ra} y_1 \left(L + y_2 \right) + \frac{E}{R} \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= -\frac{\beta}{m} y_3 - \frac{k}{m} y_2 - \frac{y_1^2}{2am} + \frac{1}{m} u_1 \end{cases}$$

2-nd step: compute the matrices

$$A = \nabla_{\boldsymbol{y}} \boldsymbol{f} \left(\boldsymbol{y}, \boldsymbol{u} \right) \Big|_{\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad B = \nabla_{\boldsymbol{u}} \boldsymbol{f} \left(\boldsymbol{y}, \boldsymbol{u} \right) \Big|_{\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad C = \nabla_{\boldsymbol{y}} \boldsymbol{g} \left(\boldsymbol{y}, \boldsymbol{u} \right) \Big|_{\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \quad D = \nabla_{\boldsymbol{u}} \boldsymbol{g} \left(\boldsymbol{y}, \boldsymbol{u} \right) \Big|_{\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}}$$

Summarizing

Linearize a nonlinear ODE around an equilibrium point

- find the equilibria
- select an equilibrium
- compute the derivatives around that equilibrium
- use the formulas
- don't forget that you are also changing the coordinate system!

Most important python code for this sub-module

This will do everything for you

https://python-control.readthedocs.io/en/latest/generated/control.linearize.html

though it is dangerous to use tools without knowing how they work

Self-assessment material

Question 1

What does it mean to linearize a nonlinear ordinary differential equation (ODE)?

Potential answers:

- I: It means approximating the nonlinear ODE with a linear model around an equilibrium point.
- II: It means replacing the ODE with a completely unrelated linear system.
- III: It means integrating the ODE analytically to find a closed-form solution.
- IV: It means ignoring all nonlinear terms in the system dynamics.
- V: I do not know

Question 2

What is the primary requirement for performing a valid linearization of a function?

Potential answers:

I: The function must be polynomial.
II: The function must be differentiable at the point of linearization.
III: The function must be bounded over the entire real line.
IV: The function must have a second derivative at all points.
V: I do not know

Question 3

Why do we typically linearize a nonlinear system around an equilibrium point?

Potential answers:

I: Because equilibrium points always yield globally valid linear models.

II: Because nonlinear systems have no real solutions.

III: Because an equilibrium point ensures the validity of the local linear approximation.

IV: Because linearization eliminates all system dynamics.

V: I do not know

Question 4

In a state-space representation of an ODE, what do the matrices A and B represent in the linearized system?

Potential answers:

I: A and B are arbitrary matrices chosen for stability.

II: A represents the second derivative of the state, and B represents the system's damping.

III: A and B are obtained by solving the system for eigenvalues and eigenvectors.

IV: A is the Jacobian of the system dynamics with respect to the state, and B is the Jacobian with respect to the input.

V: I do not know

Question 5

Which of the following is a common limitation of linearizing a nonlinear system?

Potential answers:

- I: The linearized model is only valid in a small neighborhood around the linearization point.
- II: The linearized model has no practical applications in control.
- III: Linearization makes the system unstable.
- IV: Linearization eliminates all dynamic behavior of the system.
- V: I do not know

Recap of sub-module "how to linearize an ODE"

- linearization requires following a series of steps (see the summary above)
- the model that one gets in this way is an approximation of the original model
- having a graphical understanding of what means what is essential to remember how to do things
- better testing a linear controller before a nonlinear one

?