Systems Laboratory, Spring 2025

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compute the equilibria of the system

Contents map

developed content units	taxonomy levels
equilibrium	u1, e1

prerequisite content units	taxonomy levels
ODE	u1, e1

Main ILO of sub-module "compute the equilibria of the system"

Compute the equilibria of an ODE by solving for stationary points

Is this in equilibrium?

Are these in equilibrium, while falling?

Equilibrium = a trajectory that is constant in time

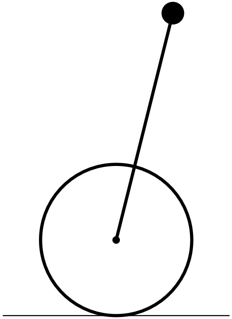
$$\dot{y}(t) = 0$$

Example

temperature of a small brick in a very large room whose temperature is 20 degrees:

$$\dot{T} = -0.5\left(T - 20\right)$$

What does it mean that this system is in equilibrium from an intuitive point of view?



What does it mean that this system is in equilibrium from a mathematical point of view?

equilibrium means $\dot{\boldsymbol{y}} = \mathbf{0}$

What does it mean that this system is in equilibrium from a mathematical point of view?

equilibrium means $\dot{\boldsymbol{y}} = \mathbf{0}$

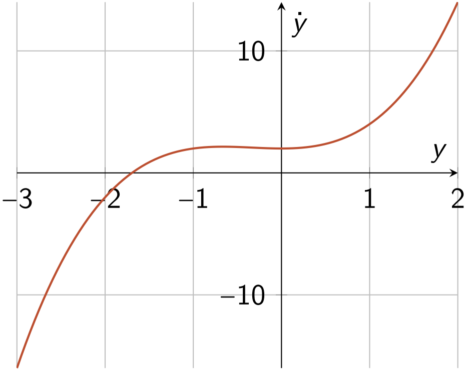
this implies

$$\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}} \text{ is an equilibrium point iff } \dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) = \mathbf{0}$$

i.e., the equilibria of a system are the zeros of $\boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right)$

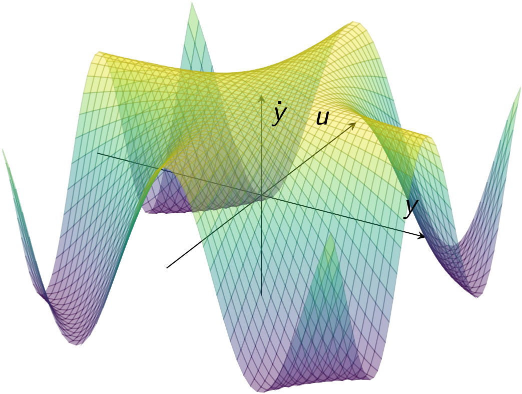
Equilibra as the zeros of $\boldsymbol{f}$, graphically

Exemplified situation of *autonomous* single output systems:



Equilibra as the zeros of $\boldsymbol{f}$, graphically

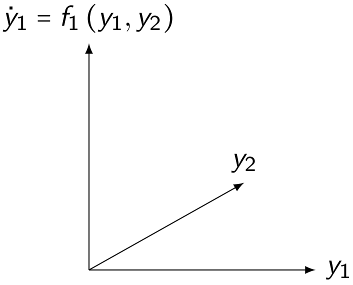
Exemplified situation of SISO (single input single output) systems:

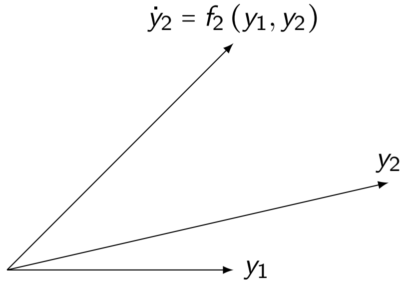


https://www.geogebra.org/classic/mmppe6hs

Equilibra as the zeros of $\boldsymbol{f}$, graphically

Exemplified situation of automonous multiple output systems:





(remember: $\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}$ equilibrium iff $\boldsymbol{f}\left(\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}\right) = \mathbf{0}$, i.e., all the components simultaneously!)

equilibria = extremely important topic in automatic control!

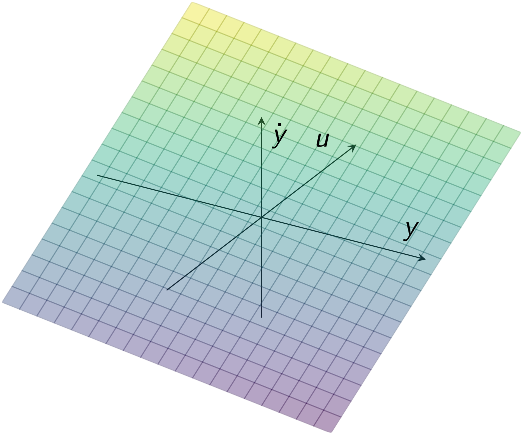
will be analysed in more details later on in this course

and much more extensively in others

(feat. Lyapunov, Krasovskii, La-Salle among others)

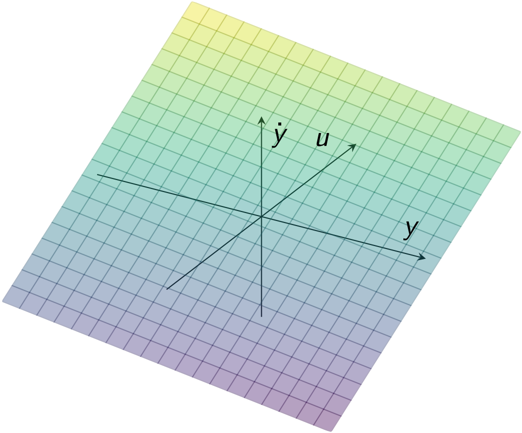
Discussion: what are the equilibria in this case?

$$\dot{y} = ay + bu \quad (1)$$



Discussion: what are the equilibria in this case?

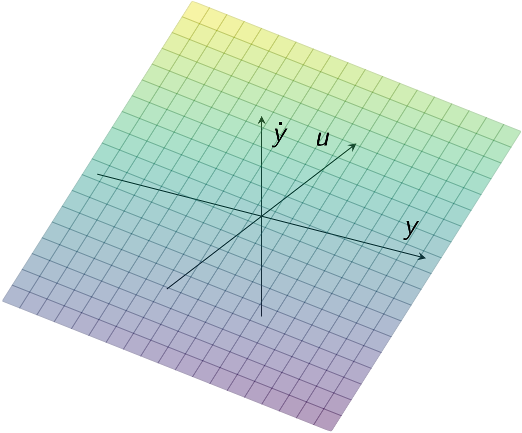
$$\dot{y} = ay + bu \tag{1}$$



$(y_0 = 0, u = 0)$ is always an equilibrium for these systems!

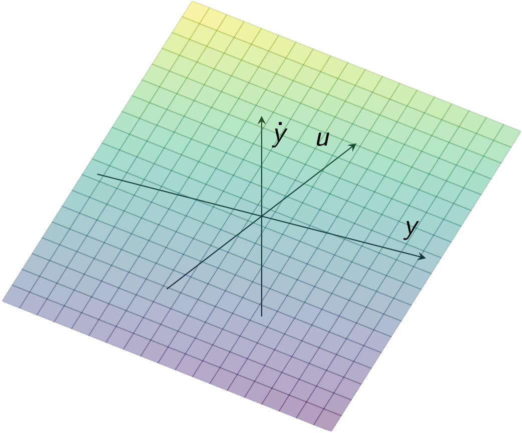
Discussion: can we have for this specific ODE an equilibrium if $u \neq \text{constant}$?

$$\dot{y} = ay + bu \tag{2}$$



Discussion: can we have for this specific ODE an equilibrium if $u \neq \mathrm{constant}$?

$$\dot{y} = ay + bu \tag{2}$$



Another example: a Lotka-Volterra model ($\neq$ real world):

$$\begin{cases} \dot{y}_{\text{rabbits}} & = \quad 0.4 \cdot y_{\text{rabbits}} - 0.5 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \\ \dot{y}_{\text{foxes}} & = -3 \cdot y_{\text{foxes}} + 0.7 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \end{cases}$$

- compute the equilibria of the system

Summarizing

Compute the equilibria of an ODE by solving for stationary points

- put $f = 0$ and compute the corresponding points. It may be that there is the need to put $u = \mathrm{constant}$

Most important python code for this sub-module

Root finding in python

https://pythonnumericalmethods.studentorg.berkeley.edu/notebooks/chapter19.05-Root-Finding-in-Python.html

Self-assessment material

Question 1

What is the mathematical definition of an equilibrium point for a dynamical system $\dot{y} = f(y, u)$?

Potential answers:

I: A point where $f(y, u)$ is maximized
II: A point where $f(y, u) = 0$
III: A point where y is always increasing
IV: A point where y is always decreasing
V: I do not know

Question 2

Which of the following statements about equilibrium points is correct?

Potential answers:

I: An equilibrium point is always stable

II: An equilibrium point is where the system's state does not change over time

III: An equilibrium point is a location where external inputs are irrelevant

IV: An equilibrium point always corresponds to $y = 0$

V: I do not know

Question 3

Graphically, how can equilibrium points be identified for an autonomous system $\dot{y} = f(y)$?

Potential answers:

I: By finding where $y = 0$

II: By finding the points where $f(y) = 0$ on the phase plot

III: By locating the steepest points of the function $f(y)$

IV: By identifying the points where y reaches its maximum or minimum values

V: I do not know

Question 4

Consider the system $\dot{T} = -0.5(T - 20)$. What is the equilibrium temperature?

Potential answers:

I: $T = 0$

II: $T = 20$

III: $T = -20$

IV: $T = 40$

V: I do not know

Question 5

For the linear system $\dot{y} = ay + bu$, under what condition is (y_0, u_0) an equilibrium?

Potential answers:

I: When $a = 0$ only
II: When $ay_0 + bu_0 = 0$
III: When $y_0 = 0$ and $u_0 = 0$ always
IV: When u_0 is arbitrary
V: I do not know

Question 6

If we have an autonomous time-varying ODE, can we have equilibria?

Potential answers:

I: No, time-variation always prevents equilibria.
II: Yes, equilibria can exist if the system allows constant solutions.
III: Only if the system is also linear.
IV: Yes, but only if the system is also periodic.
V: I do not know

Question 7

Can we have dynamical systems that do not have any equilibria?

Potential answers:

I: No, every system has at least one equilibrium.
II: Yes, with no fixed points may lack equilibria.
III: Only non-autonomous systems can lack equilibria.
IV: No, because every system must have at least a trivial equilibrium.
V: I do not know

Question 8

If we have a non-autonomous ODE, can we have equilibria if the input is always changing, e.g., $u = \sin(t)$?

Potential answers:

I: Yes, the input does not affect equilibrium conditions.
II: No, because a changing input continuously affects system states.
III: Only if the input has a zero mean.
IV: Yes, but only if the system is linear.
V: Yes, and the system does not need to be linear.
VI: I do not know

Question 9

If we have a non-autonomous LTI ODE, can we have equilibria if the input is always changing, e.g., $u = \sin(t)$?

Potential answers:

I: Yes, because LTI systems always have equilibria.
II: No, because the continuously varying input prevents a steady state.
III: Only if the system has no damping.
IV: Yes, but only if the input is periodic.
V: I do not know

Recap of sub-module "compute the equilibria of the system"

- Equilibria in dynamical systems correspond to points where the system's state does not change over time.
- Autonomous time-varying ODEs can have equilibria, but their location may vary with time.
- Some dynamical systems may not have equilibria, particularly if they involve unbounded growth.
- Non-autonomous LTI ODEs can have equilibria only if the input $u(t)$ remains constant over time.

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