Complex numbers - introduction

Contents map

developed content units	**taxonomy levels**
complex numbers	u1, e1

prerequisite content units	**taxonomy levels**
real numbers	u1, e1

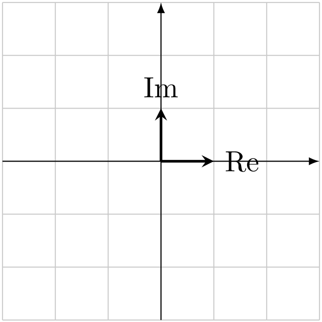
Roadmap

- definition
- sum, subtraction, multiplication, division

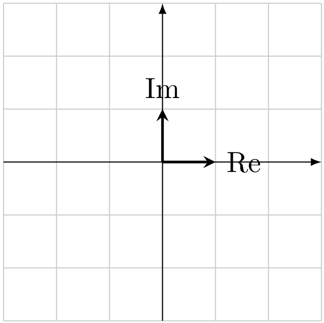
What is a complex number, and why did we introduce them?

In essence:

1. a point in the Cartesian plane
2. to be sure to find all the roots of polynomials *(i.e., be able to write polynomials in convenient forms)*

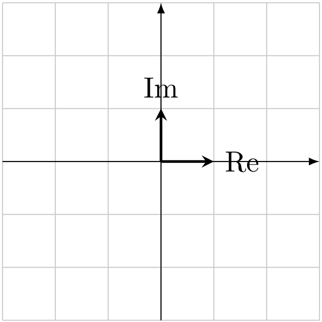


The “imaginary unit”



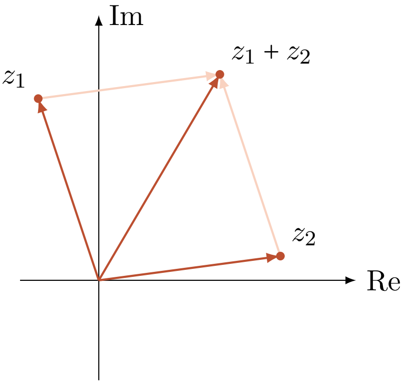
$$i \quad : \quad i^2 = -1$$

The absolute value of a complex number

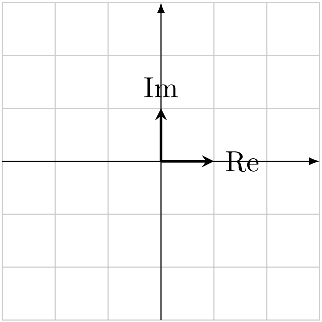


Meaning: Euclidean length of the vector. Very important for control, since very often we compute the absolute value of a transfer function at a specific $s = i\omega$ *(and very very often the transfer function is rational)*

Simple operations with complex numbers: sums, graphically



Simple operations with complex numbers: sums, mathematically

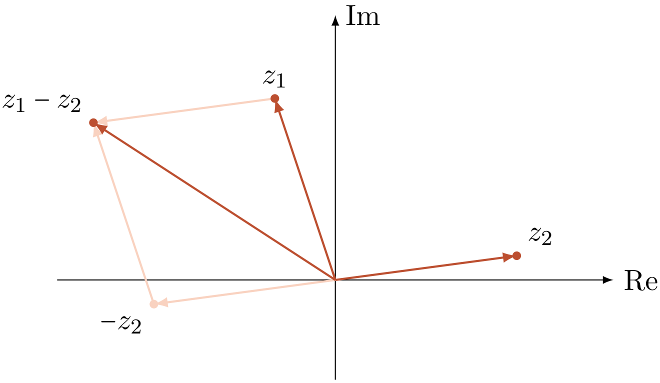


$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

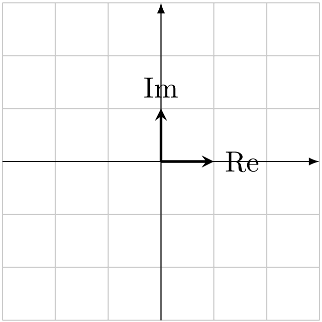
implies

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i\,(b_1 + b_2)$$

Simple operations with complex numbers: subtractions, graphically



Simple operations with complex numbers: subtractions, mathematically

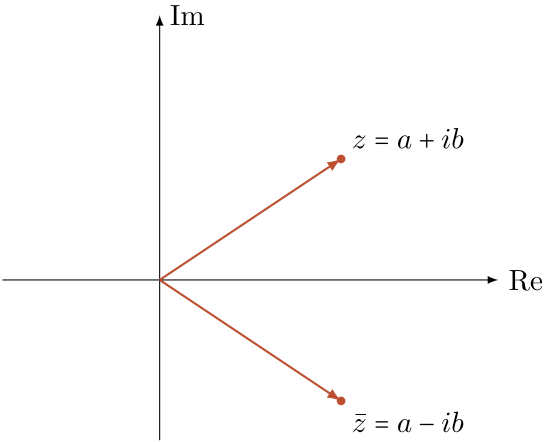


$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

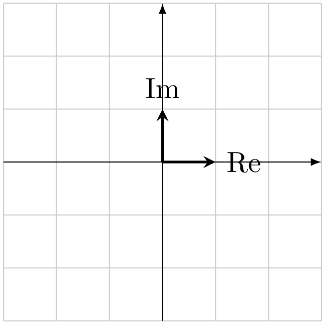
implies

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i\,(b_1 - b_2)$$

Simple operations with complex numbers: conjugation, graphically



Simple operations with complex numbers: conjugation, mathematically

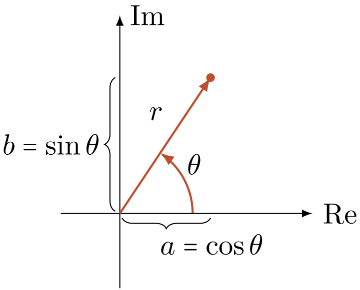


$$z_1 = a_1 + ib_1$$

implies

$$\overline{z_1} = a_1 - ib_1$$

Polar coordinates



$$z = a + ib$$

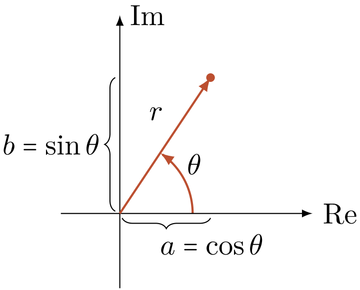
can be rewritten through r and θ so that

$$a = r\cos\theta \text{ and } b = r\sin\theta$$

so that

$$z = r\left(\cos\theta + i\sin\theta\right)$$

Polar coordinates



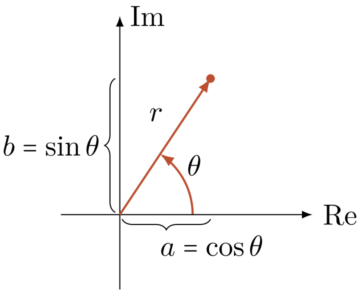
Equations:

$$
\begin{aligned}
r &= |z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}} \\
\theta &= \arg z = \operatorname{atan}(b, a) = \tan^{-1}\left(\frac{b}{a}\right)
\end{aligned}
$$

Notation:

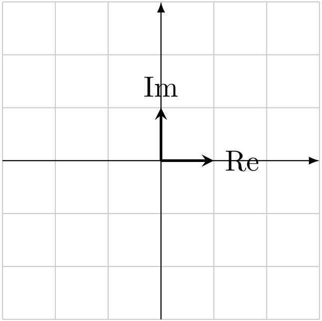
- $r =$ absolute value or modulus of z
- $\theta =$ argument, angle, or phase of z

Problem: different θ's lead to the same z



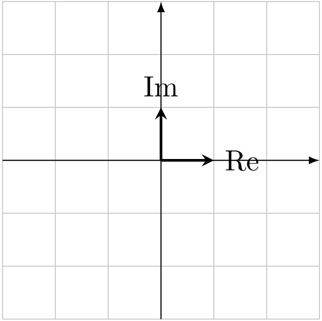
Definition: principal value of z = that value of θ that is in $[-\pi, \pi]$

Simple operations with complex numbers: multiplication (using polar coordinates)



$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i \sin\left(\theta_1 + \theta_2\right)\right]$$

Simple operations with complex numbers: multiplication (using Cartesian coordinates)



$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Conjugacy: a good way of simplifying the previous operations

- addition: $z + \overline{z} = a + ib + a - ib = 2a$, thus $\mathrm{Re}\,(z) = \dfrac{1}{2}\,(z + \overline{z})$
- subtraction: $z - \overline{z} = a + ib - a + ib = 2ib$, thus $\mathrm{Im}\,(z) = \dfrac{1}{2i}\,(z + \overline{z})$
- multiplication: $z\overline{z} = (a + ib)\,(a - ib) = a^2 + b^2$, thus $|z|^2 = z\overline{z}$

Usefulness of the multiplication: it enables Taylor expansions!

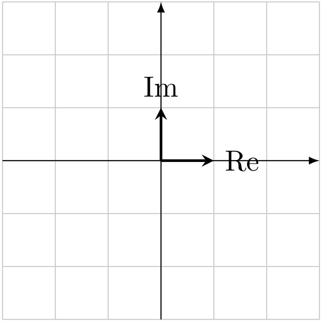
Taylor expansions: a tool to do not underestimate

$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i \sin\left(\theta_1 + \theta_2\right)\right] \quad \Longrightarrow \quad z^n \text{ well defined}$$

E.g., thus

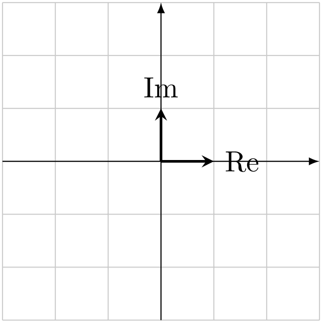
$$e^z = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \dots$$

Simple operations with complex numbers: inversion (using polar coordinates)



$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

Simple operations with complex numbers: inversion (using Cartesian coordinates)

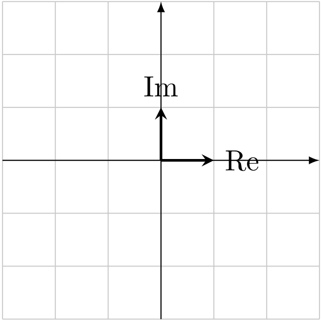


$$z_1 = a_1 + ib_1$$

implies

$$z_1^{-1} = \frac{a_1}{a_1^2 + b_1^2} - i\frac{b_1}{a_1^2 + b_1^2}$$

Simple operations with complex numbers: division (using Cartesian coordinates)



$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}$$

Recap of the module "Complex numbers - introduction"

1. there are a few operations with complex numbers that one should know how to handle
2. it will be clear later on how these operations are essential building blocks for designing filters
3. *multiplying complex numbers means multiplying the modulus and summing the phases; dividing means dividing the modulus and subtracting the phases*

Complex functions

Contents map

developed content units	taxonomy levels
complex functions	u1, e1

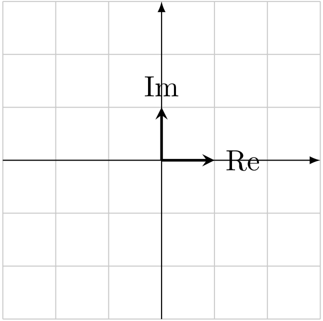
prerequisite content units	taxonomy levels
complex numbers	u1, e1

Roadmap

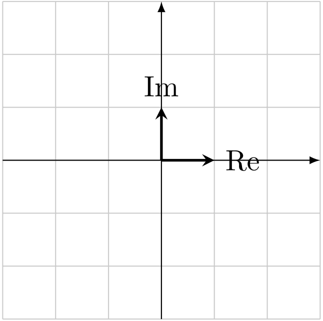
- definition
- why are they important?

Complex function: definition

$$f : \mathbb{C} \mapsto \mathbb{C} \qquad f(z) = u(x,y) + iv(x,y)$$



$\mapsto$



In polar representations: $(r, \theta) \mapsto \left(r', \theta'\right)$ with in general both r' and θ' functions of both r and θ

Example: if $f(z) = z^2 + 3z$ then what is $f(1+3j)$?

$$\begin{aligned} f(z) &= (x+iy)(x+iy) + 3x + 3iy \\ &= x^2 + 2ixy - y^2 + 3x + 3iy \\ &= x^2 - y^2 + 3x + i(2xy + 3y) \end{aligned}$$

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thus

$$
\begin{aligned}
u(x, y) &= x^2 - y^2 + 3x \\
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thus

$$\begin{aligned} u(x,y) &= x^2 - y^2 + 3x \\ v(x,y) &= 2xy + 3y \end{aligned}$$

thus

$$\begin{aligned} f(1+3j) &= u(1,3) + iv(1,3) \\ &= 1^3 - 3^2 + 3 + i(2 \cdot 1 \cdot 3 + 3 \cdot 3) \\ &= -5 + 15i \end{aligned}$$

Complex functions: why are they important?

Spoiler: the forced evolution is given by

$$Y(s) = H(s)U(s)$$

with $H(s)$ very often a ratio of complex polynomials $\Longrightarrow$ *essential tool for automatic control people: finding the zeros of complex polynomials*

Finding the zeros of complex polynomials $\implies$ finding the roots of complex functions

Finding the zeros of complex polynomials $\Longrightarrow$ finding the roots of complex functions

Primary definition: root of a complex number

if $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then the n complex roots of z are the n complex numbers $z_0, \ldots, z_{n-1}$ for which $z_k^n = z$

Finding the zeros of complex polynomials $\Longrightarrow$ finding the roots of complex functions

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How to find them? We know that

$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i \sin\left(\theta_1 + \theta_2\right)\right]$$

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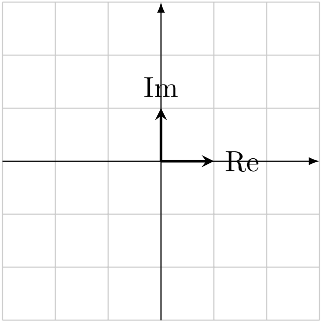
$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i \sin\left(\theta_1 + \theta_2\right)\right]$$

thus

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \quad \text{for} \quad k = 0, 1, \ldots, n-1$$

Zeros of complex functions = roots of complex numbers

Geometrically:

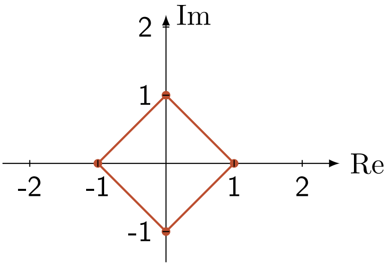


these n roots always exist

Roots of complex numbers, example: quartic roots of 1

$$\sqrt[4]{1} = \{1, i, -1, -i\}$$

(note that only two of them are in $\mathbb{R}$)



IMPORTANT: ONE SHOULD CONSIDER THE PRINCIPAL VALUE

...otherwise one may artificially add $2\pi k$ to the phase of $w = \sqrt[n]{z}$ and have an infinite number of roots ...

Why are we using so much time on this?

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Because we often have to do with objects of the type $z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$, thus we need to know what we are dealing with!

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Because we often have to do with objects of the type $z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$, thus we need to know what we are dealing with! *Essential results:*

- n-order polynomials have always from 0 to n real roots (potentially with their own multiplicities, e.g., $(z-3)^4$
- n-order polynomials have always n complex roots (again, potentially with their own multiplicities, e.g., $(z-i)^2(z+i)^2$)

Example of finding the zeros of a complex function

$$z^4 - 6iz^2 + 16 = 0$$

implies

$$z_1 = 2 + 2i \quad z_2 = -2 - 2i \quad z_3 = -1 + i \quad z_4 = 1 - i$$

(to get the solution let $y = z^2$*, and then do a bit of massaging)*

Recap of the module "Complex functions"

1. finding the zeros of complex polynomials is very important (will be shown to be an essential step in characterizing control systems)
2. the n-th roots of a complex number is a set of n complex numbers with opportune modulus and phase, so that they are placed in a geometrically balanced way along a circle in the complex plane

Complex exponentials

Contents map

developed content units	**taxonomy levels**
complex exponential	u1, e1

prerequisite content units	**taxonomy levels**
complex numbers	u1, e1
complex functions	u1, e1

Roadmap

- intuitions
- definition
- Euler's identities
- complex logarithms

In the previous episodes . . .

- complex sums and multiplications
- complex roots
- complex polynomials

→ generalizing everything, even the functions

Discussion

why are exponentials important in control?

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Because they are the essence of the modes of LTI systems, and LTI systems are often good approximations of nonlinear systems around their equilibria

First usefulness of complex exponentials: simplify notation even further

Path: rewrite

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots$$

in a notationally simpler way using $z = r(\cos\theta + i\sin\theta)$ (and, of course, using Euler's formula)

Why does Euler's formula work? (so that one may remember it more. . .)

Starting point:

$$e^z = e^{x+iy} = e^x e^{iy}$$

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but

$$e^{iy} = 1 + iy + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \ldots + \frac{1}{k!}(iy)^k + \ldots$$

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thus

$$e^z = e^x \left(\cos y + i \sin y\right)$$

Important equivalence

$$e^z = e^x\left(\cos y + i\sin y\right)$$

implies

$$\left(\cos\theta + i\sin\theta\right) = e^{i\theta}$$

The new representation given by Euler's formula and polar representations

$$z = x + iy = r\left(\cos\theta + i\sin\theta\right) \qquad r = \sqrt{x^2 + y^2} \qquad \theta = \operatorname{atan}\frac{y}{x}$$

implies

$$z = re^{i\theta}$$

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Examples

$$ze^{i\alpha} \quad = re^{i\theta}e^{i\alpha} = re^{i(\theta+\alpha)}$$

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that, by the way, implies $(x + iy)\, i = -y + ix$, i.e., a 90-degrees rotation

How to remember the trigonometric identities

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Starting point:

$$e^{iy} = \underbrace{\left(1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \frac{1}{6!}y^6 + \dots\right)}_{=\cos(y)} + i\underbrace{\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \frac{1}{7!}y^7 + \dots\right)}_{=\sin(y)}$$

(must be in this way, because "cos" is even, "sin" is odd).

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thus

$$\sin y = \frac{1}{2i}\left(e^{iy} - e^{-iy}\right) \qquad \cos y = \frac{1}{2}\left(e^{iy} + e^{-iy}\right)$$

Some important implications

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- $e^{i\pi} = -1$, $e^{\pi i/2} = i$, $e^{-\pi i/2} = -i$, $e^{-\pi i} = -1$

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- $e^{i\pi} = -1$, $e^{\pi i/2} = i$, $e^{-\pi i/2} = -i$, $e^{-\pi i} = -1$
- exponentials are never equal to 0, i.e., $e^z \neq 0$ independently of z

Some important implications

- $e^{i\pi} = -1$, $e^{\pi i/2} = i$, $e^{-\pi i/2} = -i$, $e^{-\pi i} = -1$
- exponentials are never equal to 0, i.e., $e^z \neq 0$ independently of z
- exponentials are periodic, i.e., $e^{z+2\pi i} = e^z$

Notation: "fundamental region of the exponential"

$-\pi < \mathrm{Im}\,(z) \leq \pi$

Multiplications and divisions through the complex functions

$$z_1 = r_1 e^{i\theta_1} \text{ and } z_2 = r_2 e^{i\theta_2}$$

imply

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Roots through the complex functions

$w = z^n$ is s.t. $w = re^{i\theta + 2\pi k}$ and is equal to

$$z_k = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$$

(note that besides $k = 0, 1, \ldots, n-1$, for other k's we get the same roots as before)

Recap of the module "Complex exponentials"

1. complex exponentials can be defined through Taylor expansions
2. they give birth to a refined polar notation for complex numbers that highlights the meaning of multiplication and division of complex numbers