Table of Contents I

notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

BIBO stability for LTI systems

notes

Contents map

developed content units	**taxonomy levels**
BIBO stability	u1, e1
absolute summability	u1, e1

prerequisite content units	**taxonomy levels**
discrete time LTI systems	u1, e1
impulse response	u1, e1
convoluiton	u1, e1

notes

Main ILO of sub-module "BIBO stability for LTI systems"

Graphically explain the connection between the BIBO stability of a discrete time LTI system and its impulse response

Find examples of bounded inputs that map into unbounded outputs for the case of discrete time LTI systems that are not BIBO stable

notes

- by the end of this module you shall be able to do this

Definition: absolute summability

$$f[k] \text{ absolutely summable iff } \sum_{k=-\infty}^{+\infty} |f[k]| < +\infty$$

notes

- take the absolute value of the signal, and integrate it

Question 1

Is the unit step function $\mathbb{1}[k]$ absolutely summable over the interval $(-\infty, \infty)$?

Potential answers:

I: **(wrong)** Yes, because it is bounded.
II: **(correct)** No, because its series over $(-\infty, \infty)$ diverges.
III: **(wrong)** Yes, because it is zero for $k < 0$.
IV: **(wrong)** I do not know.

Solution 1:

The unit step function is not absolutely summable because its series over $(-\infty, \infty)$ is infinite. Specifically, $\sum_{k=-\infty}^{+\infty} |\mathbb{1}[k]| = \sum_{k=0}^{+\infty}$, which diverges.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

Is the signal $x[k] = 2^k$ for $k \geq 0$, $x[k] = 0$ for $k < 0$, absolutely summable?

Potential answers:

I: **(correct)** No, because it grows exponentially as $k \to +\infty$.
II: **(wrong)** Yes, because it is a decaying exponential.
III: **(wrong)** Yes, because $\sum_{k=0}^{+\infty} |2^k|$ is finite.
IV: **(wrong)** No, because it is not defined for $k < 0$.
V: **(wrong)** I do not know.

Solution 1:

The signal $x[k] = 2^k$ is not absolutely summable because summing 2, 4, 8, etc leads to infinity.

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

Is the signal $x[k] = \sin[k]$ absolutely summable over $(-\infty, \infty)$?

Potential answers:

I: **(wrong)** Yes, because it is periodic.
II: **(correct)** No, because $\sum_{k=-\infty}^{+\infty} |\sin[k]|$ diverges.
III: **(wrong)** Yes, because its amplitude is bounded.
IV: **(wrong)** No, because it is not a decaying signal.
V: **(wrong)** I do not know.

Solution 1:

The signal $x[k] = \sin[k]$ is not absolutely summable because $\sum_{k=-\infty}^{+\infty} |\sin[k]|$ diverges. The series of the absolute value of a sinusoid over an infinite interval is infinite.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

Is the signal $x[k] = k \cdot 0.5^k$ for $k \geq 0$, $x[k] = 0$ for $k < 0$, absolutely summable?

Potential answers:

I: **(wrong)** No, because it grows linearly as $k \to \infty$.
II: **(wrong)** Yes, because it is a product of a linear function and a decaying exponential.
III: **(correct)** Yes, because $\sum_{k=0}^{+\infty} |k \cdot 0.5^k|$ is finite.
IV: **(wrong)** No, because it is not symmetric about $k = 0$.
V: **(wrong)** I do not know.

Solution 1:

The signal $x[k] = k0.5^k$ is absolutely summable.

notes

- see the associated solution(s), if compiled with that ones :)

Theorem: an LTI system is BIBO stable if and only if its impulse response is absolutely summable

BIBO stability:

$$|u[k]| < \gamma_u \quad \Longrightarrow \quad |y[k]| < \gamma_y$$

Forced response:

$$y_{\text{forced response}}[k] = h * u[k] = \sum_{\kappa=0}^{k} h[\kappa] u[k-\kappa]$$

$$\text{if} \sum_{k=-\infty}^{+\infty} |h[\kappa]| < +\infty \text{ then BIBO stability}$$

notes

- let's prove this theorem
- these two are the two definitions of BIBO stability and forced response for an LTI with impulse response h
- the fact is then that if we consider $|y[k]| \leq \left| \sum_{\kappa=0}^{k} h[\kappa] u[k-\kappa] \right|$, inserting in it the inequality $|u[k]| < \gamma_u$ we then get immediately $|y[k]| < \gamma_u \left| \sum_{\kappa=0}^{k} h[\kappa] \right|$
- this also has a geometric explanation, as done in the video of the lecture

Examples: sinusoids

https://www.youtube.com/playlist?list=PL4mJLdGEHNvhCuPXsKFrnD7AaQBlMEB6a

notes

- see the videos of these examples

Examples: any non-absolutely summable non-negative impulse response

Assumption:

$$h[k] \text{ is so that } \sum_{\kappa=-\infty}^{k} h[k]dt \xrightarrow{k\to+\infty} +\infty$$

implication:

$$\text{step} * h[k] = y_{\text{forced}}[k] = \sum_{\kappa=-\infty}^{k} h[k]dt \xrightarrow{k\to+\infty} +\infty :$$

notes

- see these examples

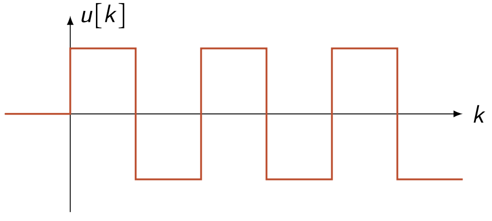
Examples: any non-absolutely summable impulse response

Assumption:

$$h[k] \text{ is so that } \sum_{\kappa=-\infty}^{k} |h[\kappa]| \xrightarrow{k\to+\infty} +\infty$$

implication:

$$\mathrm{PWM} * h[k] = y_{\mathrm{forced}}[k] = \sum_{\kappa=-\infty}^{k} h[\kappa] \xrightarrow{k\to+\infty} +\infty :$$



notes

- see the videos of these examples made in class

Summarizing

- **BIBO stable LTI system** = LTI whose impulse response is absolutely summable
- **asymptotically stable LTI system** = LTI with all its equilibria asymptotically stable
- **marginally stable LTI system** = LTI with all its equilibria marginally stable
- **unstable LTI system** = LTI with all its equilibria unstable

all the equilibria of a given LTI system are equal in nature

notes

- so, summarizing, until now we gave the following names to the following situations
- interestingly enough, an LTI cannot have one equilibrium that is unstable and another one that is marginally stable. All of them have the same properties
- this is NOT true of nonlinear systems - for example a pendulum has two equilibria of different nature

Examples: are these systems BIBO stable, marginally stable, or unstable?

- $\dot{y} = ay + bu$ with $|a| = 1$
- $\dot{y} = ay + bu$ with $|a| < 1$
- $\dot{y} = ay + bu$ with $|a| > 1$

notes

- the first is marginally stable but not BIBO stable, the second is BIBO stable and asymptotically stable, the third is unstable

And what about nonlinear systems?

more complicated! You will treat this using small gain theory, in more advanced courses

notes

- so for now we are doing LTI systems. Further on you will see also in general settings what are the conditions for BIBO stability

Other related concepts

Different types of *system* stability:

- **asymptotic input-output (system) stability:** independently of $u[k]$, $x[k] \to 0$ when $k \to +\infty$
- **marginal (or simply input-output) (system) stability:** as soon as $|u[k]| < \gamma_u$, $|x[k]| < \gamma_x$ when $k \to +\infty$
- **(system) instability:** there exists at least one signal $u[k]$ for which we cannot do the bound $|x[k]| < \gamma_x$ when $k \to +\infty$

notes

- there are then some other definitons that relate to the BIBO concept we saw until now
- it should be now clear how to generalize the concepts to the various definitions, so that if you really understood the concepts then there is no need to memorize to what the names refer to - you can "reconstruct" their meaning

But why do we do this?

it is necessary to know about potential instabilities, because our control system must stabilize them

notes

- remember also that what we are doing now is to characterize a system in terms of whether it has some potential situations for which it may explode

Summarizing

Graphically explain the connection between the BIBO stability of a discrete time LTI system and its impulse response

Find examples of bounded inputs that map into unbounded outputs for the case of discrete time LTI systems that are not BIBO stable

- check the absolute value of the impulse response
- if its series is infinite, then find an input that when convolved with the original impulse response, the result gives asymptotically the absolute value version of that impulse response

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

No dedicated python libraries for this ...

... but one can use the `control` library for checking the properties of the transfer function or impulse response of the system, if LTI of course

notes

- also this, to the best of our knowledge

Self-assessment material

notes

Question 5

Which of the following statements is true regarding the BIBO stability of an LTI system?

Potential answers:

I: **(wrong)** A system is BIBO stable if its impulse response is periodic.
II: **(correct)** A system is BIBO stable if and only if its impulse response is absolutely summable.
III: **(wrong)** A system is BIBO stable if and only if all its eigenvalues have negative real parts.
IV: **(wrong)** A system is BIBO stable if its impulse response is non-negative.
V: **(wrong)** I do not know.

Solution 1:

- BIBO stability for LTI systems 2

A system is BIBO stable if and only if its impulse response is absolutely summable, meaning $\sum_{k=-\infty}^{+\infty} |h[k]| dt < +\infty$. This ensures that bounded inputs produce bounded outputs.

notes

- see the associated solution(s), if compiled with that ones :)

Question 6

Which of the following impulse responses corresponds to a BIBO stable system?

Potential answers:

I: **(wrong)** $h[k] = e^k$ for $t < 0$, $h[k] = 0$ for $t \geq 0$.
II: **(wrong)** $h[k] = \sin[k]$.
III: **(correct)** $h[k] = e^{-k}\mathrm{step}[k]$, where $\mathrm{step}[k]$ is the unit step function.
IV: **(wrong)** $h[k] = \frac{1}{1+k^2}$ for all t.
V: **(wrong)** I do not know.

Solution 1:

The impulse response $h[k] = e^{-k}\mathrm{step}[k]$ is absolutely summable because $\sum_{k=0}^{+\infty} e^{-k} dt = 1$, which is finite, ensuring BIBO stability.

- BIBO stability for LTI systems 3

notes

- see the associated solution(s), if compiled with that ones :)

Question 7

A system has an impulse response $h[k]$ such that $\sum_{k=-\infty}^{+\infty} |h[k]| dt$ diverges. What does this imply?

Potential answers:

I: **(wrong)** The system is asymptotically stable.
II: **(correct)** The system is not BIBO stable.
III: **(wrong)** The system has a finite impulse response (FIR).
IV: **(wrong)** The system must have at least one pole in the right-half plane.
V: **(wrong)** I do not know.

Solution 1:

If the impulse response is not absolutely summable, then the system is not BIBO stable. This means there exist bounded inputs that produce unbounded outputs.

notes

- see the associated solution(s), if compiled with that ones :)

Question 8

Consider an LTI system with impulse response $h[k] = \frac{1}{1+t^2}$. What can be said about its BIBO stability?

Potential answers:

I: **(correct)** The system is BIBO stable because its impulse response is absolutely summable.
II: **(wrong)** The system is not BIBO stable because its impulse response is not causal.
III: **(wrong)** The system is not BIBO stable because its impulse response is not exponentially decaying.
IV: **(wrong)** The system is marginally stable.
V: **(wrong)** I do not know.

Solution 1:

The function $h[k] = \frac{1}{1+t^2}$ is absolutely summable since $\sum_{k=-\infty}^{+\infty} \frac{1}{1+t^2} dt = \pi$, which

notes

- see the associated solution(s), if compiled with that ones :)

Question 9

Which of the following statements correctly describes a BIBO unstable system?

Potential answers:

I: **(wrong)** A BIBO unstable system has a stable impulse response.
II: **(wrong)** A BIBO unstable system has a bounded output for every bounded input.
III: **(wrong)** A BIBO unstable system has a finite impulse response.
IV: **(correct)** A BIBO unstable system has at least one bounded input that produces an unbounded output.
V: **(wrong)** I do not know.

Solution 1:

A BIBO unstable system has at least one bounded input that produces an unbounded output. This occurs when the impulse response is not absolutely summable.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of the module "BIBO stability for LTI systems"

- for LTI systems BIBO stability is equivalent to the absolute summability of the impulse response
- for ARMA systems BIBO stability is equivalent to having the impulse response so that all its exponential terms are vanishing in time
- for nonlinear systems one shall use more advanced tools that will be seen in later on courses

notes

- in this module what we saw is thus this