Table of Contents I

- this is the table of contents of this document; each section corresponds to a specific part of the course

explain what BIBO stability means

Contents map

developed content units	taxonomy levels
unbounded signal	u1, e1
bounded signal	u1, e1
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prerequisite content units	taxonomy levels
RR	u1, e1

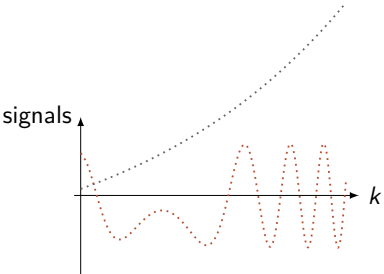## Main ILO of sub-module "explain what BIBO stability means"

Graphically explain the definition of BIBO (Bounded-Input Bounded-Output) stability and its connection to system behavior

Give examples of systems that are BIBO stable or not, and motivate their properties with physical intuitions

- by the end of this module you shall be able to do this

Definitions: bounded and unbounded signals

bounded: I can find an M for which $|y[k]| < M$ for every k. **Unbounded** otherwise

- by the end of this module you shall be able to do this

Intuition
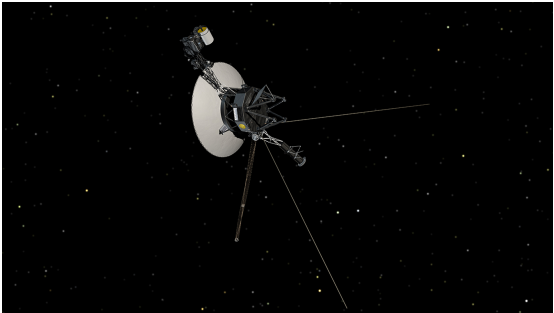
simplified model = single integrator: $v^+ = v + bu$

$$u[k] = \text{const.} \neq 0 \quad \Longrightarrow \quad \lim_{t \to +\infty} v[k] = ?$$

- if you make the spacecraft keep accelerating, its velocity will become that of light
- this is an example where even if we have a bounded input, we get an unbounded output

Interesting difference

$$v^+ = v + bu \implies \text{constant non null } u[k] \text{ causes } v[k] \text{ unbounded}$$

$$v^+ = av + bu, \; |a| < 1 \implies \text{constant non null } u[k] \text{ causes } v[k] \text{ bounded}$$

- so let's consider a simple single integrator. It is obvious that $v[k]$ is going to be the integral of $bu[k]$, so the integral of a constant signal is the area formed that a sort of rectangle with an infinitely long basis, i.e., infinite, thus $v[k]$ diverges
- at the same time if we have a first order system, this does not happen
- indeed the corresponding impulse response is an opportune $\alpha\beta^k$, and thus (one can verify this immediately also from the definition of convolution) $y[k]$ will tend to $\overline{u}$ times the integral of $\alpha\beta^k$ between 0 and $+\infty$

Visualizing this with avocados

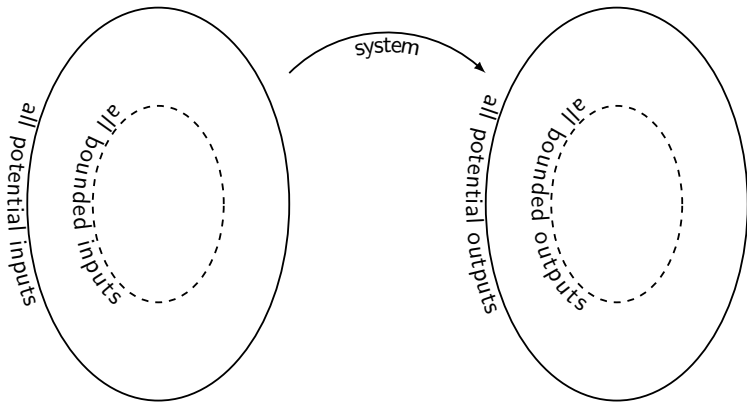for $v^+ = v + bu$ some bounded $u[k]$'s lead to unbounded $v[k]$'s
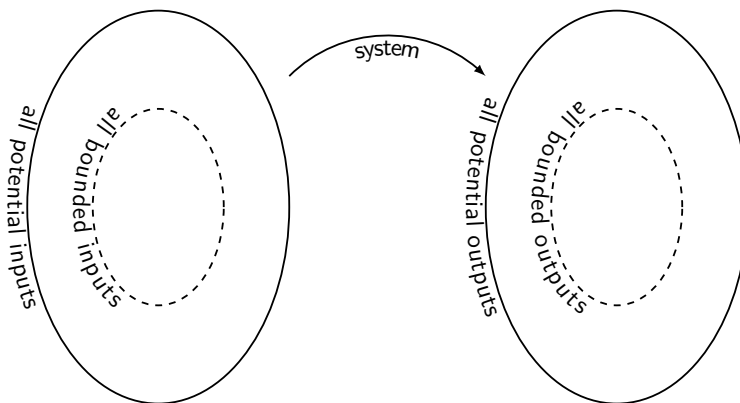
all potential inputs · all bounded inputs · system · all potential outputs · all bounded outputs

- we now see that some inputs get mapped into unbounded forced responses

Visualizing this with avocados

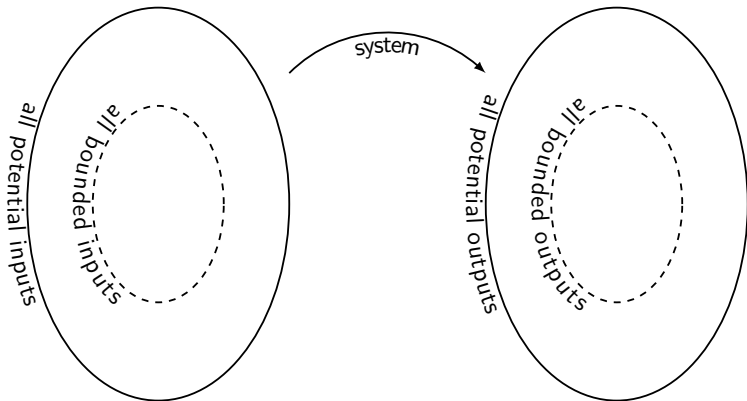for $v^+ = v + bu$ not all the bounded $u[k]$'s lead to unbounded $v[k]$'s, though

all potential inputs
all bounded inputs
system
all potential outputs
all bounded outputs

- but some inputs get mapped into bounded forced responses

Visualizing this with avocados

for $v^+ = av + bu$, $|a| < 1$ all the bounded $u[k]$'s lead to bounded $v[k]$'s

all potential inputs
all bounded inputs
system
all potential outputs
all bounded outputs

- and in this case all the bounded inputs get mapped into bounded forced responses

Important difference: the two systems have two different "stability" properties relative to how inputs get mapped into outputs!

- indeed the two systems are intrinsically different

Remark

- "Stability" referring to specific equilibria:
  - simply stable equilibrium
  - convergent equilibrium
  - asymptotically stable equilibrium

- "Stability" referring to specific systems:
  - Bounded Input Bounded Output (BIBO) stable systems (we are seeing this now)
  - Input to State Stable (ISS) systems (we will not see this in this module)

- more precisely be careful that there are several concepts of stability, these five being the most common ones in automatic control
- we will not see input to state stability, that you will instead see when you will do nonlinear systems in more details than what we do here

Definition: BIBO stability

If $\boldsymbol{y}^+ = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})$ is so that if $\boldsymbol{u}$ is bounded then $\boldsymbol{y}$ will be for sure bounded, then that system is said BIBO stable
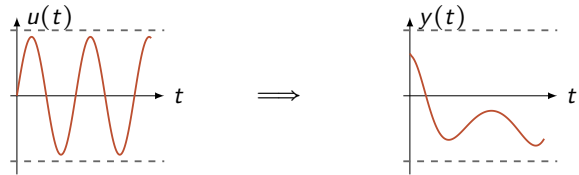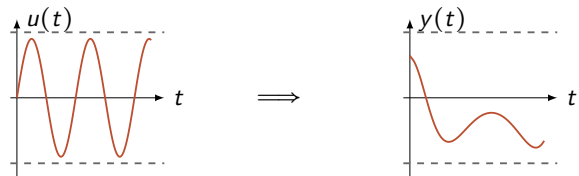
- so let's consider a generic system, also a MIMO one, it does not matter
- what we saw in the previous examples is that sometimes a bounded input causes an unbounded output, while in others a bounded input causes only bounded outputs
- the mathematical definition of this property is then quite simple: consider an input, also time-varying. Are you able to find a bound to the input, i.e., say that it is max TOT and min TOT? If so, then the output shall also be limited, i.e., be max TOT2 and min TOT2 (note that these bounds may be different, and depending on the specific input, of course). If for any input that is bounded we know that the output will be bounded then the system is BIBO
- if I though find just one bounded input that produces an unbounded output, then the system is NOT BIBO

Definition: BIBO stability

If $\boldsymbol{y}^+ = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})$ is so that there exists at least one $\boldsymbol{u}$ bounded for which the corresponding $\boldsymbol{y}$ will be unbounded, then that system is said **not** BIBO stable

Example

Is $y^+ = 2y + u$ BIBO stable?

- no, and we can say this immediately because this is an unstable system

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Example

Is $y^+ = yu$ BIBO stable?

- no, and we can say this immediately because this contains an integrator when $u = 1$

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Example

Is $y^+ = u$ BIBO stable?

- yes, it is just a shift of the u signal

Example

Is $y^+ = -y$ BIBO stable?

- no, this is not even non-autonomous! we cannot define the input output behavior here

Summarizing

> **Graphically explain** the definition of BIBO (Bounded-Input Bounded-Output) stability and its connection to system behavior

> **Give examples** of systems that are BIBO stable or not, and motivate their properties with physical intuitions

- if starting from the kernel of the input-avocado we are guaranteed to end up in the kernel of the output-avocado, then we have BIBO stability
- if not, not
- BIBO stability is a concept that relates to non-autonomous system - and it is not necessary that it it LTI, may also be nonlinear. But it needs to have at least one input

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

No dedicated python libraries for this . . .

. . . but one can use the `control` library for checking the properties of the transfer function or impulse response of the system, if LTI of course

- also this, to the best of our knowledge

Self-assessment material

Question 1

Which of the following statements best describes BIBO (Bounded-Input Bounded-Output) stability?

Potential answers:

I: **(wrong)** A system is BIBO stable if all outputs remain constant regardless of the input.

II: **(correct)** A system is BIBO stable if every bounded input leads to a bounded output.

III: **(wrong)** A system is BIBO stable if it has at least one bounded output for an unbounded input.

IV: **(wrong)** A system is BIBO stable if it is asymptotically stable.

V: **(wrong)** I do not know

Solution 1:
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BIBO stability means that for any bounded input, the system output remains bounded. This definition does not require the system to be asymptotically stable.

Question 2

Which of the following systems is NOT BIBO stable?

Potential answers:

I: **(correct)** $y^+ = -2y + u$.

II: **(correct)** $y^+ = y + u$.

III: **(correct)** $y^+ = -y + 3u$.

IV: **(wrong)** $y^+ = -0.5y + u$.

V: **(wrong)** I do not know

Solution 1:

A discrete time LTI system is NOT BIBO stable if the coefficient in front of the y is not smaller than one in absolute value.

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Question 3

Which graphical interpretation correctly illustrates a system that is NOT BIBO stable?

Potential answers:

I: **(wrong)** A system where all bounded inputs correspond to bounded outputs.

II: **(correct)** A system where at least one bounded input results in an unbounded output.

III: **(wrong)** A system where all unbounded inputs lead to unbounded outputs.

IV: **(wrong)** A system where the impulse response is always decreasing over time.

V: **(wrong)** I do not know

Solution 1:

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A system that is NOT BIBO stable has at least one bounded input that results in an unbounded output, violating the BIBO stability condition.

Question 4

A spacecraft is modeled as a single integrator $v^+ = v + bu$. What can be said about its BIBO stability?

Potential answers:

I: **(correct)** The system is NOT BIBO stable because a constant nonzero input leads to an unbounded velocity.

II: **(wrong)** The system is BIBO stable because velocity is a smooth function of time.

III: **(wrong)** The system is BIBO stable because acceleration remains bounded.

IV: **(wrong)** The system is BIBO stable because it eventually reaches a steady-state velocity.

V: **(wrong)** I do not know

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Solution 1:

Since the velocity is the integral of the input, a constant nonzero input results in

Question 5

Can one define the BIBO properties of an autonomous system?

Potential answers:

  I: **(correct)** No, BIBO stability is defined in terms of input-output behavior.
 II: **(wrong)** Yes, every system has a well-defined BIBO stability property.
III: **(wrong)** Only if the system is linear and time-invariant.
 IV: **(wrong)** Yes, but only for discrete-time systems.
  V: **(wrong)** I do not know

Solution 1:

BIBO stability concerns the boundedness of outputs given bounded inputs. Autonomous systems do not have external inputs, so BIBO stability is not directly applicable.

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Question 6

If a system has two inputs, can it be BIBO stable with respect to one input but not the other?

Potential answers:

  I: **(correct)** Yes, different inputs can excite different dynamics in the system.
 II: **(wrong)** No, BIBO stability is a system-wide property.
III: **(wrong)** Only if the system is nonlinear.
 IV: **(wrong)** Only for discrete-time systems.
  V: **(wrong)** I do not know

Solution 1:

BIBO stability is defined per input-output pair. A system can be stable for one input if that input does not excite unstable modes, while another input may lead to unbounded output.

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Question 7

BIBO stability properties are connected to equilibrium points, so if a system has three equilibria, do we need to analyze BIBO stability separately for each equilibrium?

Potential answers:

   I: **(correct)**   No, BIBO stability is an input-output property and does not depend on equilibria.
  II: **(wrong)**   Yes, because each equilibrium defines a different stability region.
 III: **(wrong)**   Only if the system is nonlinear.
 IV: **(wrong)**   Only for continuous-time systems.
  V: **(wrong)**   I do not know

Solution 1:

BIBO stability is defined for the entire systems input-output response, not per equilibrium. Stability around an equilibrium is a different concept (e.g., Lyapunov stability).

- see the associated solution(s), if compiled with that ones :)

Question 8

Is an integrator a BIBO stable system?

Potential answers:

   I: **(correct)**   No, an integrator accumulates input over time, leading to unbounded output for bounded input.
  II: **(wrong)**   Yes, because the output remains predictable.
 III: **(wrong)**   Only in discrete-time systems.
 IV: **(wrong)**   Only if the initial condition is zero.
  V: **(wrong)**   I do not know

Solution 1:

An integrator produces an output that grows indefinitely in response to a constant or oscillatory input, meaning it is not BIBO stable.

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module "explain what BIBO stability means"

- BIBO stability means "a bounded input must imply a bounded output"
- it is a concept that in general it is disconnected to that of marginal stability / convergence of an equilibrium

- the most important remarks from this sub-module are these ones