Systems Laboratory, Spring 2025

Damiano Varagnolo – CC-BY-4.0

notes

- welcome to the course!
- on this side of this document you will find notes that accompany the text typically visualized in class
- these notes are meant to convey the messages that are not displayed in the text on the side, and basically constitute what the teacher intends to say in class

Table of Contents I

notes

- this is the table of contents of this document; each section corresponds to a specific part of the course when is linearizing meaningful

notes

Contents map

developed content units	**taxonomy levels**
linearization	u1, e1

prerequisite content units	**taxonomy levels**
ODE	u1, e1

notes

Main ILO of sub-module "when is linearizing meaningful"

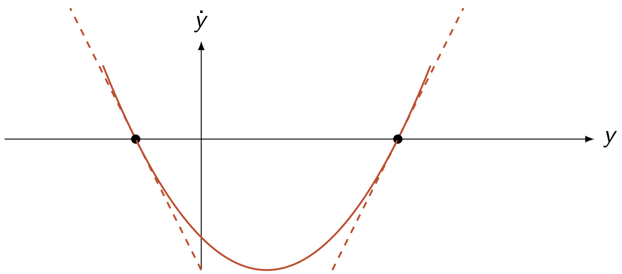
Assess the validity of the approximation introduced when linearizing a nonlinear ODE around an equilibrium point

Evaluate the meaning and applicability of linearization in different contexts, discussing when it provides a reasonable approximation and when it does not

notes

- by the end of this module you shall be able to do this

Discussion: around which equilibrium may we consider this model approximation a "good one"?

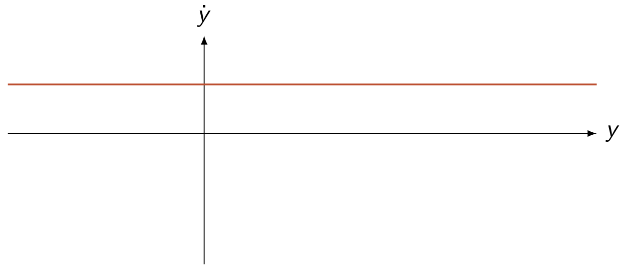


also for the 'unstable' equilibrium the approximation may be a good one - depends on the time horizon under consideration and how close y_0 is to the equilibrium

notes

- the fact that A and $\boldsymbol{f}$ are different (even if one is the approximation of the other one) means that, if we think at their physical meaning, starting from the same point the two models give different indications towards where $\boldsymbol{y}$ should go
- this means that the trajectories will be different
- how much different, though? Depends of course on some sort of distance between A and $\boldsymbol{f}$
- for the asymptotically stable equilibrium the approximation will get better and better in time; for the unstable equilibrium worse and worse in time
- recall though that one may consider an arbitrarily small neighborhood of the approximation point. In this way one may think that the linearized version may be an arbitrarily good approximation, if one focuses in a sufficiently small neighborhood
- in this course we will not see how to compute bounds of the error between these two trajectories; you will do it in later on courses

Discussion: is it always meaningful to linearize?

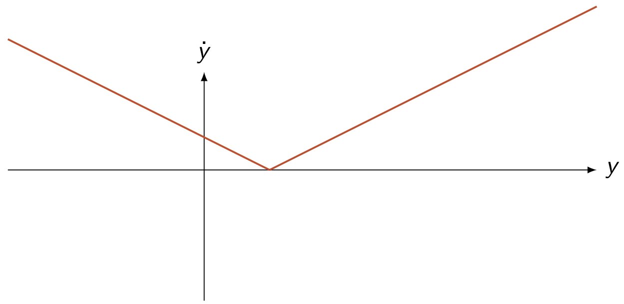


in this case we do not have equilibria

notes

- a simple case that shows that if we do not have equilibria actually we can't linearize

Discussion: and here, can we linearize?

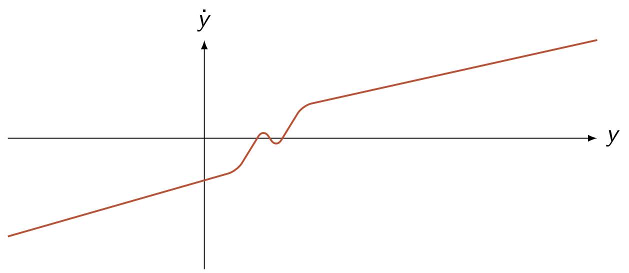


in this case we cannot compute the first derivative

notes

- a simple case that shows that if we do not have continuity for the derivative actually we can't linearize

Discussion: can we trust the stable linearized system for this case?



in this case the basin of attraction is very small

notes

- a simple case that shows that if the basin of attractions are very small, the linearized system may be trusted in a very small region

Summarizing

Assess the validity of the approximation introduced when linearizing a nonlinear ODE around an equilibrium point

Evaluate the meaning and applicability of linearization in different contexts, discussing when it provides a reasonable approximation and when it does not

- if we have an asymptotically stable equilibrium, the approximation improves in time
- if we have an unstable equilibrium, the approximation degrades in time
- the closer we start from the equilibrium, the better
- the bigger the curvature of the ODE, the more "local" the results will be

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

This will do everything for you

`https://python-control.readthedocs.io/en/latest/generated/control.linearize.html`

though it is dangerous to use tools without knowing how they work

notes

- be always wary of using code without knowing the effects and meaning of the operations they do

Self-assessment material

notes

Question 1

When linearizing a nonlinear ODE around an equilibrium point, which of the following conditions ensures that the approximation improves over time?

Potential answers:

I: **(wrong)** The equilibrium point is unstable.
II: **(correct)** The equilibrium point is asymptotically stable.
III: **(wrong)** The ODE has a high curvature near the equilibrium point.
IV: **(wrong)** The initial point is far from the equilibrium.
V: **(wrong)** I do not know.

Solution 1:

The approximation improves over time when the equilibrium point is asymptotically stable. This is because trajectories near such equilibria converge toward the equilibrium, making the linearized model increasingly accurate.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

In which of the following cases is it NOT meaningful to linearize a nonlinear ODE?

Potential answers:

I: **(wrong)** The ODE has multiple equilibrium points.
II: **(correct)** The ODE does not have any equilibrium points.
III: **(wrong)** The ODE has a small basin of attraction.
IV: **(wrong)** The ODE is highly nonlinear.
V: **(wrong)** I do not know.

Solution 1:

Linearization is not meaningful when the ODE does not have any equilibrium points, as the process of linearization relies on approximating the system near an equilibrium.

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

Which of the following factors limits the validity of a linearized ODE approximation?

Potential answers:

I: **(wrong)** The linearized system has a stable equilibrium.
II: **(correct)** The basin of attraction of the equilibrium is very small.
III: **(wrong)** The ODE is continuous and differentiable.
IV: **(wrong)** The initial point is close to the equilibrium.
V: **(wrong)** I do not know.

Solution 1:

A very small basin of attraction limits the validity of the linearized approximation, as the region where the approximation holds becomes very restricted.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

What happens to the accuracy of a linearized ODE approximation near an unstable equilibrium point over time?

Potential answers:

- I: **(correct)** The approximation degrades over time.
- II: **(wrong)** The approximation improves over time.
- III: **(wrong)** The accuracy remains constant.
- IV: **(wrong)** The accuracy depends on the curvature of the ODE.
- V: **(wrong)** I do not know.

Solution 1:

Near an unstable equilibrium, the approximation degrades over time because trajectories diverge from the equilibrium, making the linearized model less accurate.

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

Which of the following statements about linearization is true?

Potential answers:

- I: **(wrong)** Linearization is always a good approximation for any nonlinear ODE.
- II: **(correct)** Linearization provides a better approximation when the initial point is closer to the equilibrium.
- III: **(wrong)** Linearization is only valid for ODEs with high curvature.
- IV: **(wrong)** Linearization cannot be applied to stable systems.
- V: **(wrong)** I do not know.

Solution 1:

Linearization provides a better approximation when the initial point is closer to the equilibrium, as the linearized model is most accurate in a small neighborhood around the equilibrium.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module "when is linearizing meaningful"

- be careful when using a linearized system - be always aware of where it comes from

notes

- the most important remarks from this sub-module are these ones