notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

Complex numbers - introduction

notes

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Contents map

developed content units	taxonomy levels
complex numbers	u1, e1

prerequisite content units	taxonomy levels
real numbers	u1, e1

notes

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Roadmap

- definition
- sum, subtraction, multiplication, division

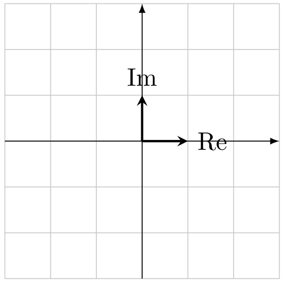
notes

- this module refreshes some operations that are essential building blocks for designing filters
- these concepts should be already known but better to refresh them before starting learning how to design filters

What is a complex number, and why did we introduce them?

In essence:

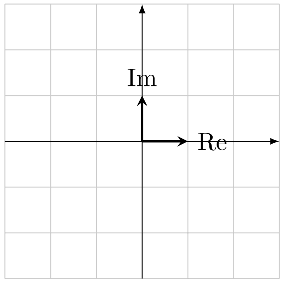
1. a point in the Cartesian plane
2. to be sure to find all the roots of polynomials *(i.e., be able to write polynomials in convenient forms)*



notes

- in its essence, complex numbers have been introduced to be able to find all the roots of polynomials
- essentially they are points in a Cartesian plane

The "imaginary unit"

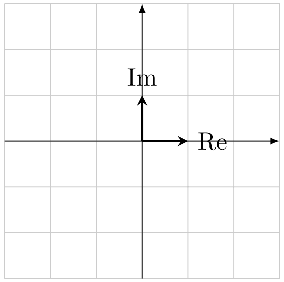


$$i \quad : \quad i^2 = -1$$

notes

- formally the imaginary unit satisfies this equality
- there is more behind, though: we recommend you to watch `https://www.youtube.com/watch?v=SP-YJe7Vldo`

The absolute value of a complex number

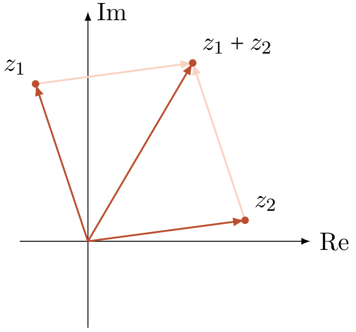


Meaning: Euclidean length of the vector. Very important for control, since very often we compute the absolute value of a transfer function at a specific $s = i\omega$ *(and very very often the transfer function is rational)*

notes

- as we will see later on, the absolute value of a complex number is a paramount operation that will be done very often
- for now in the course the concept of "transfer function" has not been introduced yet, but you will see how taking the absolute value of a transfer function is something that is done very often and that involves taking the absolute value of complex numbers

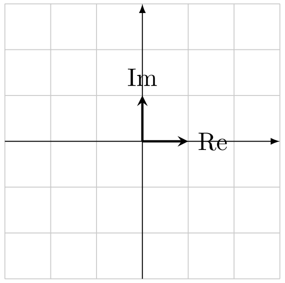
Simple operations with complex numbers: sums, graphically



notes

- the sum of two complex numbers can be thought graphically as the typical sum of two vectors

Simple operations with complex numbers: sums, mathematically



$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

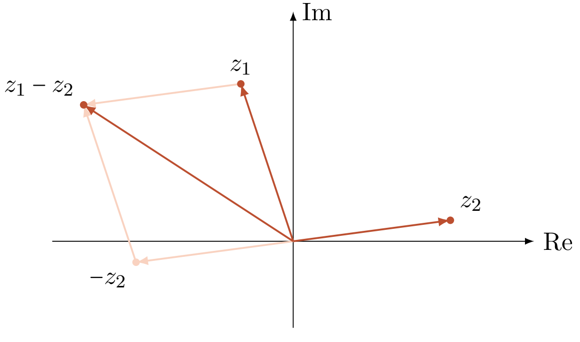
implies

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i\,(b_1 + b_2)$$

notes

- mathematically speaking the formulas are then these ones

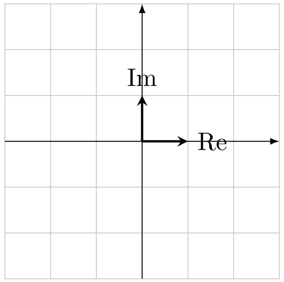
Simple operations with complex numbers: subtractions, graphically



notes

- similarly, the subtraction is as subtracting vectors

Simple operations with complex numbers: subtractions, mathematically



$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

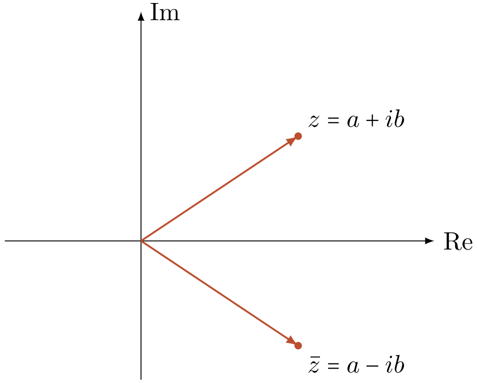
implies

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i\,(b_1 - b_2)$$

notes

- that mathematically translates into these formulas

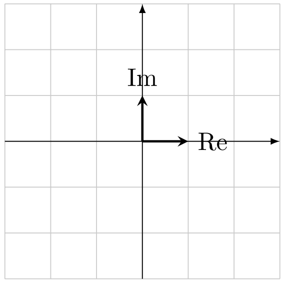
Simple operations with complex numbers: conjugation, graphically



notes

- conjugating means flipping along the real axis
- the importance of conjugating will be clear when we introduce the product of two complex numbers
- moreover remember that the roots of a real polynomial are either real or complex conjugate pairs

Simple operations with complex numbers: conjugation, mathematically



$$z_1 = a_1 + ib_1$$

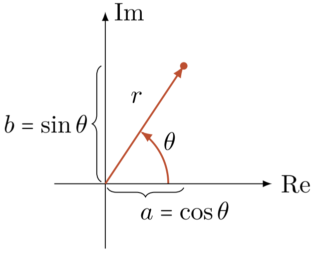
implies

$$\overline{z_1} = a_1 - ib_1$$

notes

- mathematically flipping along the real axis means this

Polar coordinates



$$z = a + ib$$

can be rewritten through r and θ so that

$$a = r\cos\theta \text{ and } b = r\sin\theta$$

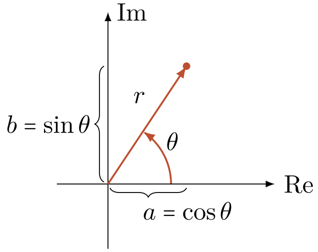
so that

$$z = r\left(\cos\theta + i\sin\theta\right)$$

notes

- instead of using classical “x-y” coordinates, one may also use polar ones
- graphically and mathematically the two are connected in this way

Polar coordinates



Equations:

$$
\begin{aligned}
r &= |z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}} \\
\theta &= \arg z = \operatorname{atan}(b, a) = \tan^{-1}\left(\frac{b}{a}\right)
\end{aligned}
$$

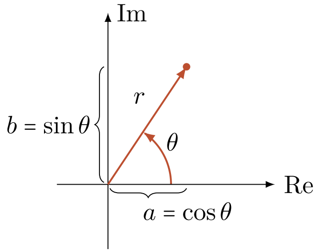
Notation:

- $r =$ absolute value or modulus of z
- $\theta =$ argument, angle, or phase of z

notes

- further mathematical quantities that can be defined using polar coordinates are these ones
- note that we still haven't introduced the multiplication among complex numbers, but this will be done soon

Problem: different θ's lead to the same z

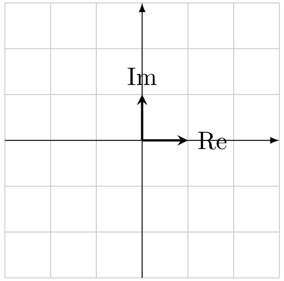


Definition: principal value of $z =$ that value of θ that is in $[-\pi, \pi]$

notes

- there is also something to be remembered, that is that polar representations are somehow not univoque, in the sense that formally different angles may mean actually the same thing
- we thus give a specific name to that angle that is the smallest one in absolute value among all the ones that give the same complex number

Simple operations with complex numbers: multiplication (using polar coordinates)

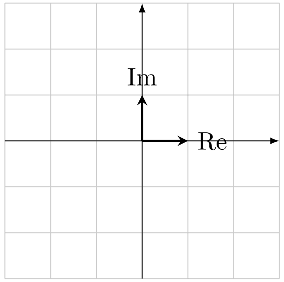


$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i \sin\left(\theta_1 + \theta_2\right)\right]$$

notes

- multiplying two complex numbers means multiplying their modulus and summing their phases
- thus it is a simultaneous stretching and rotating
- see also the video in `https://www.khanacademy.org/math/precalculus/x9e81a4f98389efdf:complex/x9e81a4f98389efdf:complex-mul/a/visualizing-complex-multiplication` for some animations

Simple operations with complex numbers: multiplication (using Cartesian coordinates)



$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$z_1 z_2 = \left(a_1 + ib_1\right)\left(a_2 + ib_2\right) = \left(a_1 a_2 - b_1 b_2\right) + i\left(a_1 b_2 + a_2 b_1\right)$$

notes

- the equations for the multiplication with the Cartesian coordinates are a bit more involved, corresponding to these

Conjugacy: a good way of simplifying the previous operations

- addition: $z + \overline{z} = a + ib + a - ib = 2a$, thus $\mathrm{Re}\,(z) = \frac{1}{2}\,(z + \overline{z})$
- subtraction: $z - \overline{z} = a + ib - a + ib = 2ib$, thus $\mathrm{Im}\,(z) = \frac{1}{2i}\,(z + \overline{z})$
- multiplication: $z\overline{z} = (a + ib)\,(a - ib) = a^2 + b^2$, thus $|z|^2 = z\overline{z}$

notes

- note that the operation of conjugating complex numbers simplifies quite a lot the operations we saw up to now

Usefulness of the multiplication: it enables Taylor expansions!

Taylor expansions: a tool to do not underestimate

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\right] \quad \Longrightarrow \quad z^n \text{ well defined}$$

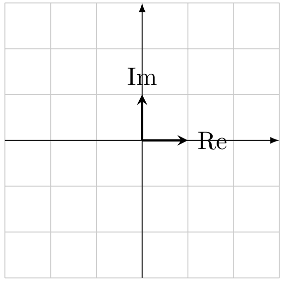
E.g., thus

$$e^z = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \ldots$$

notes

- note that multiplying complex numbers enables doing several other operations
- one of the most important ones for our course is taking Taylor expansions
- thus multiplication is a fundamental building block

Simple operations with complex numbers: inversion (using polar coordinates)

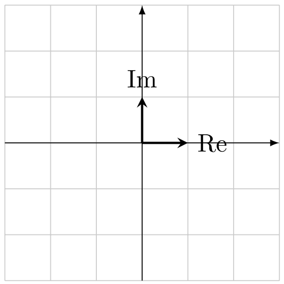


$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

notes

- inverting z means finding that complex number z' that if we multiply the two the product is $1 + i0$
- this means finding that z' that has the inverse of the modulus and the opposite of the phase

Simple operations with complex numbers: inversion (using Cartesian coordinates)



$$z_1 = a_1 + ib_1$$

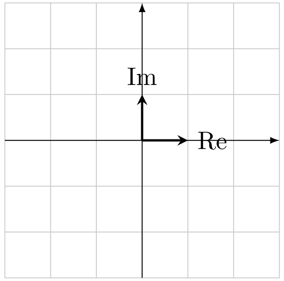
implies

$$z_1^{-1} = \frac{a_1}{a_1^2 + b_1^2} - i\frac{b_1}{a_1^2 + b_1^2}$$

notes

- once again with Cartesian coordinates the formulas look more involved
- graphically, though, they are the same thing

Simple operations with complex numbers: division (using Cartesian coordinates)



$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2$$

implies

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i \frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}$$

notes

- this thus enables also dividing complex numbers
- from a graphical perspective it means dividing the modulus and subtracting the phase

Recap of the module "Complex numbers - introduction"

1. there are a few operations with complex numbers that one should know how to handle
2. it will be clear later on how these operations are essential building blocks for designing filters
3. *multiplying complex numbers means multiplying the modulus and summing the phases; dividing means dividing the modulus and subtracting the phases*

notes

- the importance of what we have been doing in this module is this one
- the essential operation that you shall remember for this course is the one highlighted her

Complex functions

notes

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Contents map

developed content units	taxonomy levels
complex functions	u1, e1

prerequisite content units	taxonomy levels
complex numbers	u1, e1

notes

▪

Roadmap

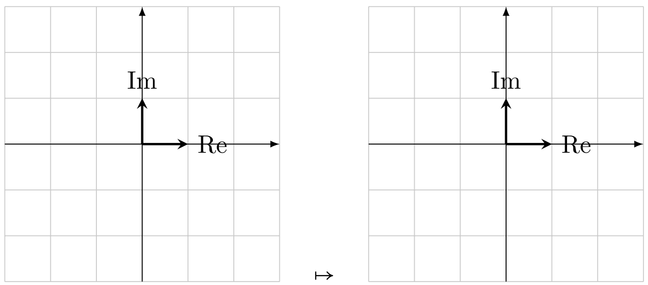
- definition
- why are they important?

notes

- in this module we go through the definition of complex functions, and then describe why they are important for this course

Complex function: definition

$$f : \mathbb{C} \mapsto \mathbb{C} \qquad f(z) = u(x,y) + iv(x,y)$$



In polar representations: $(r, \theta) \mapsto (r', \theta')$ with in general both r' and θ' functions of both r and θ

notes

- a complex function is a map from the complex plane to the complex plane
- it can be written both in terms of Cartesian maps or polar maps
- in any case a complex function is a warping of the 2D space

Example: if $f(z) = z^2 + 3z$ then what is $f(1+3j)$?

$$\begin{aligned} f(z) &= (x+iy)(x+iy) + 3x + 3iy \\ &= x^2 + 2ixy - y^2 + 3x + 3iy \\ &= x^2 - y^2 + 3x + i(2xy + 3y) \end{aligned}$$

thus

$$\begin{aligned} u(x,y) &= x^2 - y^2 + 3x \\ v(x,y) &= 2xy + 3y \end{aligned}$$

thus

$$\begin{aligned} f(1+3j) &= u(1,3) + iv(1,3) \\ &= 1^3 - 3^2 + 3 + i(2 \cdot 1 \cdot 3 + 3 \cdot 3) \\ &= -5 + 15i \end{aligned}$$

notes

- just to give an example, to operate with complex functions means handling both imaginary and real components as done here

Complex functions: why are they important?

Spoiler: the forced evolution is given by

$$Y(s) = H(s)U(s)$$

with $H(s)$ very often a ratio of complex polynomials $\Longrightarrow$ *essential tool for automatic control people: finding the zeros of complex polynomials*

notes

- for control purposes we need to be able to deal with some specific complex functions (more precisely, the so called transfer functions, that we will analyse soon)
- through these functions we will be able to compute forced evolutions

Finding the zeros of complex polynomials $\Longrightarrow$ finding the roots of complex functions

Primary definition: root of a complex number

if $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then the n complex roots of z are the n complex numbers $z_0, \ldots, z_{n-1}$ for which $z_k^n = z$

How to find them? We know that

$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i \sin\left(\theta_1 + \theta_2\right)\right]$$

thus

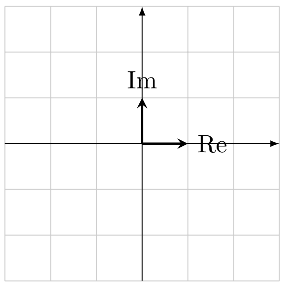
$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i \sin\frac{\theta + 2k\pi}{n}\right) \quad \text{for} \quad k = 0, 1, \ldots, n-1$$

notes

- to know what it means to find the zeros of a complex polynomial requires knowing first what it means to find the roots of a complex function
- and to know what these are, one needs to know first what are the roots of a complex number
- the definition is mutuated directly from the one that is used for real numbers. The n-th roots of z are all that numbers that elevated to that power they give n
- how to find them? Well, we know that multiplying two complex numbers means multiplying the modulus and summing the phases
- then the n-th roots must be such that this happens
- in other words, the modulus of the root must have a modulus that is a (real) root, and the phase of the root must be s.t. if multiplied by n it gives the phase of the original number
- watch out though at the fact that the polar coordinates have a 2π periodicity - this means that there will be n different roots, each with the same modulus but a different phase

Zeros of complex functions = roots of complex numbers

Geometrically:



these n roots always exist

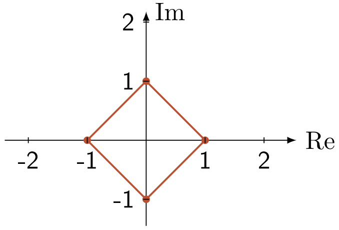
notes

- geometrically thus these roots are placed on a circle of radius $\sqrt[n]{r}$ and at n different regular angular positions
- see also `https://www.youtube.com/watch?v=N0Y8ia57C24&ab_` for more information, if this is not clear

Roots of complex numbers, example: quartic roots of 1

$$\sqrt[4]{1} = \{1, i, -1, -i\}$$

(note that only two of them are in $\mathbb{R}$)



notes

- applying this concept, we see that the quartic roots of unity are these ones

IMPORTANT: ONE SHOULD CONSIDER THE PRINCIPAL VALUE

. . . otherwise one may artificially add $2\pi k$ to the phase of $w = \sqrt[n]{z}$ and have an infinite number of roots . . .

notes

- and be careful for this. . .

Why are we using so much time on this?

Because we often have to do with objects of the type $z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$, thus we need to know what we are dealing with! *Essential results:*

- n-order polynomials have always from 0 to n real roots (potentially with their own multiplicities, e.g., $(z-3)^4$
- n-order polynomials have always n complex roots (again, potentially with their own multiplicities, e.g., $(z-i)^2(z+i)^2$)

notes

- and this is a motivational slide, with some results one should always remember
- important: if one sums up the various multiplicities of the various complex roots then one gets n

Example of finding the zeros of a complex function

$$z^4 - 6iz^2 + 16 = 0$$

implies

$$z_1 = 2 + 2i \quad z_2 = -2 - 2i \quad z_3 = -1 + i \quad z_4 = 1 - i$$

(to get the solution let $y = z^2$, and then do a bit of massaging)

notes

- an example is this one

Recap of the module "Complex functions"

1. finding the zeros of complex polynomials is very important (will be shown to be an essential step in characterizing control systems)
2. the n-th roots of a complex number is a set of n complex numbers with opportune modulus and phase, so that they are placed in a geometrically balanced way along a circle in the complex plane

notes

- these are then the most important messages of this module

Complex exponentials

notes

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Contents map

developed content units	**taxonomy levels**
complex exponential	u1, e1

prerequisite content units	**taxonomy levels**
complex numbers	u1, e1
complex functions	u1, e1

notes

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Roadmap

- intuitions
- definition
- Euler's identities
- complex logarithms

notes

- in this module we see a quantity whose importance is high for control systems engineers

In the previous episodes . . .

- complex sums and multiplications
- complex roots
- complex polynomials

→ generalizing everything, even the functions

notes

- and this module follows the trend initiated in the previous modules of generalizing as much as possible things from the 'real' realm to the 'complex' one

Discussion

why are exponentials important in control?

Because they are the essence of the modes of LTI systems, and LTI systems are often good approximations of nonlinear systems around their equilibria

notes

- an important result that was discovered in the linear algebra part is that exponentials are essential quantities characterizing LTI systems
- since eigenvalues may be complex, we need to be able to deal with complex exponentials

First usefulness of complex exponentials: simplify notation even further

Path: rewrite

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots$$

in a notationally simpler way using $z = r(\cos\theta + i\sin\theta)$ (and, of course, using Euler's formula)

notes

- the first goal of this module is to rewrite a complex exponential, that can always be defined by means of its Taylor expansion, in a notationally convenient way

Why does Euler's formula work? (so that one may remember it more...)

Starting point:

$$e^z = e^{x+iy} = e^x e^{iy}$$

but

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \dots + \frac{1}{k!}(iy)^k + \dots \\ &= \underbrace{\left(1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \frac{1}{6!}y^6 + \dots\right)}_{=\cos(y)} + i\underbrace{\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \frac{1}{7!}y^7 + \dots\right)}_{=\sin(y)} \end{aligned}$$

thus

$$e^z = e^x(\cos y + i\sin y)$$

notes

- here there are some formulas, all based on Taylor expansions of the various involved functions, that lead to an interesting rewriting of e^z
- note how here using Cartesian coordinates is more appealing than polar ones

Important equivalence

$$e^z = e^x (\cos y + i \sin y)$$

implies

$$(\cos\theta + i\sin\theta) = e^{i\theta}$$

notes

- this equivalence will be used very soon, together with its implications

The new representation given by Euler's formula and polar representations

$$z = x + iy = r(\cos\theta + i\sin\theta) \qquad r = \sqrt{x^2 + y^2} \qquad \theta = \operatorname{atan}\frac{y}{x}$$

implies

$$z = re^{i\theta}$$

This confirms the intuition that multiplying z in the complex plane by $e^{i\theta}$ means rotating z of θ radiants *anti-clockwise* in $\mathbb{C}$

Examples

$$\begin{aligned} ze^{i\alpha} &= re^{i\theta}e^{i\alpha} = re^{i(\theta+\alpha)} \\ zi &= re^{i\theta}e^{i\frac{\pi}{2}} = re^{i\left(\theta+\frac{\pi}{2}\right)} \end{aligned}$$

that, by the way, implies $(x + iy)\, i = -y + ix$, i.e., a 90-degrees rotation

notes

- indeed the previous equivalence in the previous slide implies that we can compact the notation for polar coordinates even more
- and with this notation we see immediately that multiplication means essentially rotation
- this means that these examples can be carried out in this way

How to remember the trigonometric identities

Starting point:

$$e^{iy} = \underbrace{\left(1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \frac{1}{6!}y^6 + \dots\right)}_{=\cos(y)} + i \underbrace{\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \frac{1}{7!}y^7 + \dots\right)}_{=\sin(y)}$$

(must be in this way, because "cos" is even, "sin" is odd). But also

$$e^{-iy} = \underbrace{\left(1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \frac{1}{6!}y^6 + \dots\right)}_{=\cos(y)} - i \underbrace{\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \frac{1}{7!}y^7 + \dots\right)}_{=-\sin(y)}$$

thus

$$\sin y = \frac{1}{2i}\left(e^{iy} - e^{-iy}\right) \qquad \cos y = \frac{1}{2}\left(e^{iy} + e^{-iy}\right)$$

notes

- in this slide there are some mnemonics, to help memorizing (that is important too, from a learning perspective)
- a way to remember cosine and sine's expansions, is that they alternate signs, and cosine starts on the x axis, where one finds the 1. The sine is thus the other 'alternation'
- putting the opposite in the exponent leads then to this expansion
- and then one may combine the two into one

Some important implications

- $e^{i\pi} = -1$, $e^{\pi i/2} = i$, $e^{-\pi i/2} = -i$, $e^{-\pi i} = -1$
- exponentials are never equal to 0, i.e., $e^z \neq 0$ independently of z
- exponentials are periodic, i.e., $e^{z+2\pi i} = e^z$

Notation: "fundamental region of the exponential"

$-\pi < \mathrm{Im}(z) \leq \pi$

notes

- a set of implications follow immediately
- the first four are essentially the quartic roots of unity, using the polar-complex notation and the geometrical interpretation of what the roots of a complex number are
- the second thing (very important) is that $e^{something}$ cannot be zero, independently of that something
- plus once again the complex exponentials inherit the periodicity of polar representations
- this means that one should always consider the principal values of the complex numbers one is handling

Multiplications and divisions through the complex functions

$$z_1 = r_1 e^{i\theta_1} \text{ and } z_2 = r_2 e^{i\theta_2}$$

imply

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

notes

- given the polar-complex notation, now performing multiplications and divisions is immediate: here the multiplication / division of the moduli and the sum / subtraction of the phases is directly visible in the expressions

Roots through the complex functions

$w = z^n$ is s.t. $w = re^{i\theta + 2\pi k}$ and is equal to

$$z_k = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$$

(note that besides $k = 0, 1, \ldots, n-1$, for other k's we get the same roots as before)

notes

- this means that one can see roots directly too

Recap of the module "Complex exponentials"

1. complex exponentials can be defined through Taylor expansions
2. they give birth to a refined polar notation for complex numbers that highlights the meaning of multiplication and division of complex numbers

notes

- the most important messages of this module are then these