FOUNDATIONS OF SIGNALS AND SYSTEMS
Solved exercises and homework assignment on:
Fourier series in MatLab

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Observe the outcome of truncated Fourier series, and appreciate the presence of the Gibbs' phenomenon at discontinuities. Practice yourself with numerically evaluated Fourier coefficients, to be able to represent any periodic signal through its (truncated) Fourier series.

Carefully read the solved exercises first and then do your homework. For each question in the homework there is a solution, which you can read after trying to solve it yourself to verify the adequacy of the method you used, as well as the correctness of your result.

Solved exercises.

Solve the following MatLab problems:

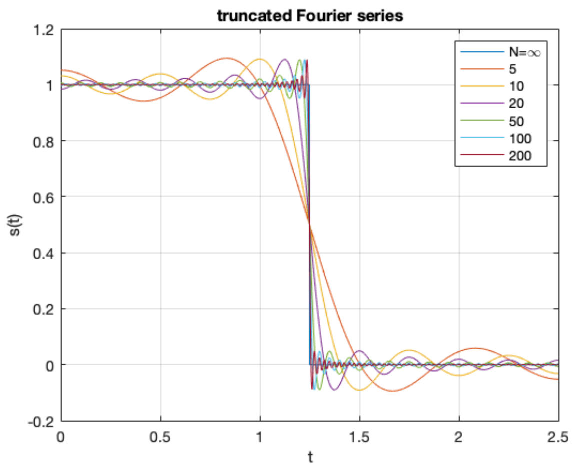
1. Consider a rectangular wave of period $T_p = 5$ and duty cycle $d = \frac{1}{2}$, and its truncated Fourier series
$$s_N(t) = \sum_{k=-N}^{N} a_k \, e^{jk\omega_0 t} \ , \quad \omega_0 = \frac{2\pi}{T_p} \ , \ a_k = d \ \mathrm{sinc}(kd) \ .$$
In the same plot, show how the truncated series approximates the square wave for $N = 5, 10, 20, 50, 100, 200$, and observe the Gibbs phenomenon in the range $[0, \frac{1}{2}T_p]$. Use a very small sampling spacing T for the representation in MatLab.

2. Consider again a rectangular wave $s(t)$ of period $T_p = 5$ and duty cycle $d = \frac{1}{2}$ and its truncated Fourier series
$$s_N(t) = \sum_{k=-N}^{N} a_k \, e^{jk\omega_0 t} \ , \quad \omega_0 = \frac{2\pi}{T_p} \ ,$$
where the Fourier coefficients are now approximated via the numerical integration
$$a_k = \frac{1}{T_p} \int_0^{T_p} s(t) e^{-jk\omega_0 t} dt \simeq b_k = \frac{1}{T_p} \cdot T \sum_{n=0}^{M-1} s(nT) \, e^{-jk\omega_0 nT} \ ,$$
for $T = T_p/M$ and a large M indicating the number of samples in the period. Compare, for $N = 100$, the different output obtained by the true coefficients a_k and the approximated coefficients b_k, using $M = 200, 500, 1000$.

Solutions.

1. In the code we first define constants and set the sampling spacing to a very small value to fully capture the Gibbs phenomenon. The different truncated series are obtained by a loop in N. Inside this loop, the truncated series is computed by first initializing a vector to zero values, and by then adding the contribution of each single Fourier coefficient, which is performed through a cycle in k. Before plotting, the imaginary part is removed, since this only accounts for numerical errors. The plots superposition is obtained by freezing the figure through an hold command.

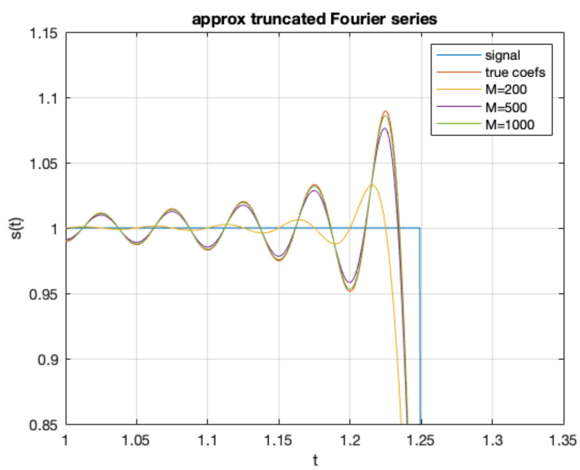
```
Tp = 5; % period
d = .5; % duty cycle
om0 = 2*pi/Tp; % omega0
T = 0.001; % sampling spacing

t = 0:T:Tp/2;
s = square_wave(t,Tp,d);
figure
plot(t,s)
grid
axis([0 Tp/2 -.2 1.2])
xlabel('t')
ylabel('s(t)')
hold on

for N = [5,10,20,50,100,200]
    tfs = zeros(size(t)); % truncated Fourier
        series
    for k = -N:N % cicle on coefficients
        ak = d*sinc(k*d); % Fourier coefficient
        tfs = tfs +  ak*exp(1i*k*om0*t);
    end
    tfs = real(tfs); % prevent numerical errors
    plot(t,tfs)
end
legend('N=\infty','5','10','20','50','100','200')
title('truncated Fourier series')

function s = square_wave(t,Tp,d)
t1 = mod(t/Tp,1);
s = rect(t1/d) + rect((t1-1)/d);
end

function s = rect(t)
s = (abs(t)<.5)+.5*(abs(t)==.5);
end
```



2. This exercise repeats the previous one in its first part, then re-evaluates the coefficients by numerical integration trough a cycle on M where samples nT and $s(nT)$ are first stored, then used for calculating the approximate coefficients through a sum. The plot is zooming on the signal part that better evidences details, to appreciate how only $M = 1000$ is able to closely match the true result.

```
Tp = 5; % period
d = .5; % duty cycle
om0 = 2*pi/Tp; % omega0
T = 0.001; % sampling spacing
N = 100; % number of Fourier coefficients

t = 0:T:Tp/2;
s = square_wave(t,Tp,d);
figure
plot(t,s)
grid
axis([1 1.3 .85 1.15])
xlabel('t')
ylabel('s(t)')
hold on

% true coefficients
tfs = zeros(size(t)); % truncated Fourier series
for k = -N:N % cicle on coefficients
    ak = d*sinc(k*d); % Fourier coefficient
    tfs = tfs +  ak*exp(1i*k*om0*t);
end
tfs = real(tfs); % prevent numerical errors
```

```
plot(t,tfs)

% numerical coefficients
for M = [200,500,1000]
    tfs = zeros(size(t)); % truncated Fourier
        series
    nT = (0:M-1)*Tp/M;
    snT = square_wave(nT,Tp,d);
    for k = -N:N % cicle on coefficients
        bk = sum(snT.*exp(-1i*k*om0*nT))/M;
        tfs = tfs +  bk*exp(1i*k*om0*t);
    end
    tfs = real(tfs); % prevent numerical errors
    plot(t,tfs)
end
legend('signal','true coefs','M=200','M=500','M
    =1000')
title('approx truncated Fourier series')

function s = square_wave(t,Tp,d)
t1 = mod(t/Tp,1);
s = rect(t1/d) + rect((t1-1)/d);
end

function s = rect(t)
s = (abs(t)<.5)+.5*(abs(t)==.5);
end
```



Homework assignment.

Solve the following MatLab problems:

1. Consider the triangular wave

$$s(t) = \operatorname*{rep}_{T_p} \operatorname{triang}(\tfrac{t}{dT_p}) \ ,$$

   period $T_p = 5$ and duty cycle $d = \frac{1}{2}$, and its truncated Fourier series

$$s_N(t) = \sum_{k=-N}^{N} a_k \, e^{jk\omega_0 t} \ , \quad \omega_0 = \frac{2\pi}{T_p} \ , \ a_k = d \operatorname{sinc}^2(kd) \ .$$

   In the same plot, show how the truncated series approximates the square wave for $N = 5, 10, 20, 50, 100, 200$ in the range $[0, \frac{1}{2}T_p]$. Can we see the Gibbs phenomenon? Use a very small sampling spacing T for the representation in MatLab.

2. Consider a periodic signal of period $T_p = 3$, defined in a period as

$$s(t) = \begin{cases} t & 0 < t < 1 \\ 1 & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$

   so that in MatLab we can have

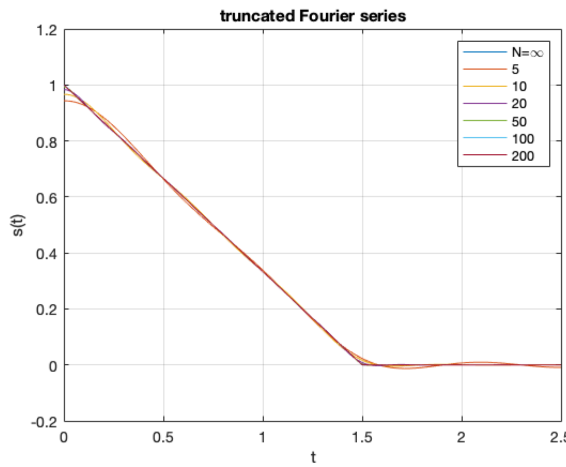
```
function s = signal(t)
t1 = mod(t,3);
s = t1.*(t1<1) + (t1>=1).*(t1<2);
end
```

   Evaluate its Fourier coefficients by resorting to numerical integration, as in solved exercise 2, with $M = 10^3$ and 10^4, and show the corresponding truncated Fourier series for $N = 100$. Is the Gibbs phenomenon visible? Where?

Solutions.

1. The code can mimic that of solved exercise 1, but since the triangular wave has no discontinuities, no Gibbs phenomenon can be observed.

```
Tp = 5; % period
d = .3; % duty cycle
om0 = 2*pi/Tp; % omega0
T = 0.001; % sampling spacing

t = 0:T:Tp/2;
s = triang_wave(t,Tp,d);
figure
plot(t,s)
grid
axis([0 Tp/2 -.2 1.2])
xlabel('t')
ylabel('s(t)')
hold on

for N = [5,10,20,50,100,200]
    tfs = zeros(size(t)); % truncated Fourier
       series
    for k = -N:N % cicle on coefficients
        ak = d*sinc(k*d)^2; % Fourier coefficient
        tfs = tfs +  ak*exp(1i*k*om0*t);
    end
    tfs = real(tfs); % prevent numerical errors
    plot(t,tfs)
end
legend('N=\infty','5','10','20','50','100','200')
title('truncated Fourier series')

function s = triang_wave(t,Tp,d)
t1 = mod(t/Tp,1);
s = triang(t1/d) + triang((t1-1)/d);
end

function s = triang(t)
s = (abs(t)<1).*(1-abs(t));
end
```



2. This exercise repeats solved exercise 2 by substituting the signal values. Note that with $M = 10^3$ we already obtain a very reliable result, showing that the Gibbs phenomenon only appears at discontinuities.

```
Tp = 3; % period
om0 = 2*pi/Tp; % omega0
T = 0.001; % sampling spacing
N = 100; % number of Fourier coefficients

t = 0:T:Tp;
s = signal(t);
figure
plot(t,s)
grid
xlabel('t')
ylabel('s(t)')
hold on

% numerical coefficients
for M = [1e3,1e4]
    tfs = zeros(size(t)); % truncated Fourier
        series
    nT = (0:M-1)*Tp/M;
    snT = signal(nT);
    for k = -N:N % cicle on coefficients
        bk = sum(snT.*exp(-1i*k*om0*nT))/M;
        tfs = tfs +  bk*exp(1i*k*om0*t);
    end
    tfs = real(tfs); % prevent numerical errors
    plot(t,tfs)
```

```
end
legend('signal','M=1000','M=10000')
title('approx truncated Fourier series')
```