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notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

state space representations

notes

Contents map

developed content units	taxonomy levels
state of a system	u1, e1
separation principle	u1, e1

prerequisite content units	taxonomy levels
RR	u1, e1

notes

Main ILO of sub-module "state space representations"

Define the meaning of "state space representation" in the context of linear and non-linear discrete time dynamical systems

notes

- by the end of this module you shall be able to do this

Discussion: which information do you need to forecast accurately how long you may use your cellphone before its battery hits 0%?

notes

- think at which factors are important for you

Summarizing

these pieces of information contain all I need to forecast the future evolution of the battery level:

- current level of charge of the battery
- how much I will use the phone in the future
- how healthy the battery of my phone is
- which environmental factors may induce additional effects (too warm, too cold)

notes

- the state condenses somehow the past
- from a control point of view this is important, because the state somehow works as a "memory": to decide wich u is best right now, I just need to check what is the current state – I do not care about what state the system was experiencing before
- so this is a concept that is very instrumental for control

A simple model of the battery charge as a dynamical system

$$\text{Time Remaining} = \frac{\text{Remaining Capacity}}{\text{Discharge Rate}} \qquad \text{example: } \frac{2000\mathrm{mAh}}{500\mathrm{mA}} = 4\text{hours}$$

rewriting as a RR:

- $y[k] = Q[k] =$ remaining battery capacity at the discrete time kT (mAh)
- $u[k] = I[k] =$ current discharge rate at the discrete time kT (mA)

$$\Longrightarrow \qquad y^{+} = y - u$$

notes

- let's make a physical model of this
- and if we make it as a LTI ODE, given that the set of phenomena that we are assuming as constituting the models do not comprise self-discharge, we shall have something that in free evolution looks like $\dot{y} = 0y = 0$
- moreover if I draw positive current I decrease the charge, thus we get a 'minus' sign associated to the u

What is a state?

$$\begin{cases} x^{+} & = -u \\ y & = x \end{cases}$$

"the current value of the state $x[k]$ contains all the information necessary to forecast the future evolution of the output $y[k]$ and of the state $x[k]$, assuming to know the future $u[k]$. I.e., to compute the future values $y[k+\kappa]$ and $x[k+\kappa]$ it is enough to know the current $x[k]$ and the current and future inputs $u[k:k+\kappa]$"

notes

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- from a control point of view this is important, because the state somehow works as a "memory": to decide wich u is best right now, I just need to check what is the current state – I do not care about what state the system was experiencing before
- so this is a concept that is very instrumental for control

Question 1

In a spring-mass system, which of the following is a valid state variable?

Potential answers:

I: **(wrong)** The temperature of the spring.
II: **(correct)** The displacement of the mass from its equilibrium position.
III: **(wrong)** The color of the mass.
IV: **(wrong)** The external force applied to the system.
V: **(wrong)** I do not know.

Solution 1:

The displacement of the mass from its equilibrium position is a valid state variable because it describes the system's configuration and is essential for predicting its future behavior. Temperature and color are irrelevant, and the external force is an input, not a state.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

Which of the following pairs of variables can fully describe the state of a spring-mass system?

Potential answers:

I: **(wrong)** The mass of the spring and the stiffness of the mass.
II: **(wrong)** The external force and the displacement of the mass.
III: **(correct)** The displacement of the mass and the velocity of the mass.
IV: **(wrong)** The acceleration of the mass and the color of the spring.
V: **(wrong)** I do not know.

Solution 1:

The displacement of the mass and the velocity of the mass fully describe the state of a spring-mass system because they capture the system's current configuration (displacement) and its rate of change (velocity). Mass, stiffness, external force, and color are not state variables.

notes

- see the associated solution(s), if compiled with that ones :)

What do we mean with “modelling a state-space dynamical system”?

Defining

$$\begin{cases} \boldsymbol{x}^{+} & = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\theta}) \\ \boldsymbol{y} & = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\theta}) \end{cases}$$

and

- the variables
  - $\boldsymbol{u}$ = the inputs
  - $\boldsymbol{d}$ = the disturbances
  - $\boldsymbol{x}$ = the states vector
  - $\boldsymbol{y}$ = the measured outputs
- the structure of the functions $\boldsymbol{f}$ and $\boldsymbol{g}$
- the value of the parameters $\boldsymbol{\theta}$

notes

- these are called state space representations
- take home message: the input-output maps saw in other modules are not the unique ways of representing systems

Discrete time state space model - definition

Ingredients:

- the number of inputs, outputs and state variables must be finite
- the difference equations must be first order
- the separation principle *(the current value of the state contains all the information necessary to forecast the future evolution of the outputs and of the state)* shall be satisfied

state space model = finite set of first-order difference equations that connect a finite set of inputs, outputs and state variables so that they satisfy the separation principle

notes

- so, if we recall what we did in some modules ago, this was the formal definition of a state space system
- remember that, first of all, it is a finite representation: for example a metal bar that is heating up, we may describe it with partial difference equations. But this would mean considering the temperature in every point, and this means an infinite number of points - no good
- we work with computers, and somehow we need always to consider a discrete and finite number of objects. Thus we consider finite number of states

State space representations - Notation

- $u_1, \ldots, u_m$ = inputs
- $x_1, \ldots, x_n$ = states
- $y_1, \ldots, y_p$ = outputs
- $d_1, \ldots, d_q$ = disturbances

notes

- remember also that this is the standard notation (but the q in d_q, for which there is no standard notation)
- if you will use something different in your job you will look like a fool

State space representations - Notation

$$\begin{aligned} \boldsymbol{x}^+ &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned}$$

- $\boldsymbol{f}$ = state transition map
- $\boldsymbol{g}$ = output map

notes

- finally, we use this notation

examples

notes

Bank Account Balance

Scenario: modeling the balance of a bank account over time with monthly deposits and interest:

- $x[k]$ = account balance at month k
- $u[k]$ = monthly deposit/withdrawing at month k
- $r[k]$ = monthly interest rate *(may be time-varying!)*

State-Space Model:

$$\begin{cases} x[k+1] = (1 + r[k]) \cdot x[k] + u[k] & \textit{(State equation)} \\ y[k] = x[k] & \textit{(Output equation)} \end{cases}$$

notes

- this is actually a state space system, since we have the 'separation principle'

Particular case: exponential growth, scalar version

$$x^+ = \alpha x + \beta u \qquad y = x$$

notes

- the previous example can be generalized in this way
- we will see better later on that "exponentials" play a big role here, since if we neglect u you see that we have that we must have that the derivative of y must be proportional to y itself. Exponentials have this property (also sinusoids, but we know that sinusoids are complex exponentials, because of Euler's identities)
- we will see this better later on though
- here note that how much y grows depends on both y and u, and this dependence is "fixed" by α and β

Exponential growth, matricial version

Generalization of all linear systems

$$\begin{cases} \boldsymbol{x}^+ &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} \end{cases}$$

notes

- this example generalizes the one saw before. Better to recall the geometrical interpretation of a matrix times a vector, that highlights each column of A to be a direction in the space where $\dot{\boldsymbol{x}}$ lives, and every component of $\boldsymbol{x}$ being thus how much that direction of that column should be followed
- the same interpretation of columns times scalars follows for the term $B\boldsymbol{u}$. Here each term of $\boldsymbol{u}$
- also for this type of ODE we will have that exponentials play a big role

Lotka-Volterra

- y_{prey} := prey
- y_{pred} := predator

$$\begin{cases} y_{\text{prey}}^{+} &= \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ y_{\text{pred}}^{+} &= -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

`./LotkaVolterraSimulator.ipynb`

notes

- of course not all the systems are of the type $\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$. This one is for example nonlinear, since there is a product among the y's that cannot be captured by the linear relation above
- this example was seen before. More information and history behind it in `https://en.wikipedia.org/wiki/Lotka%E2%80%93Volterra_equations`
- go through the python notebook for more information

Summarizing

Define the meaning of "state space representation" in the context of linear and non-linear discrete time dynamical systems

- recall the definition of state space model
- be sure to have interiorized the separation principle with some practical examples

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

Important library

https://python-control.readthedocs.io/en/0.10.1/conventions.html#state-space-systems

notes

- this is more or less a reference library

Self-assessment material

notes

Question 3

In the battery charge model $y^+ = y - u$, what does y represent?

Potential answers:

I: **(wrong)** The discharge rate of the battery.
II: **(correct)** The remaining battery capacity.
III: **(wrong)** The temperature of the battery.
IV: **(wrong)** The external force applied to the battery.
V: **(wrong)** I do not know.

Solution 1:

In the battery charge model, y represents the remaining battery capacity. The equation $y^+ = y - u$ describes how the remaining capacity decreases based on the discharge rate u.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

What does the separation principle in state-space models imply?

Potential answers:

I: **(wrong)** The system must have an infinite number of states.
II: **(wrong)** The system must be linear.
III: **(correct)** The current state contains all information needed to predict future outputs and states.
IV: **(wrong)** The system must have no inputs or disturbances.
V: **(wrong)** I do not know.

Solution 1:

The separation principle states that the current state $x[k]$ contains all the information necessary to forecast the future evolution of the outputs $y[k]$ and the state $x[k]$, given the future inputs $u[k]$.

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

In the bank account model $x[k+1] = (1 + r[k]) \cdot x[k] + u[k]$, what does $u[k]$ represent?

Potential answers:

I: **(wrong)** The interest rate at month k.
II: **(correct)** The monthly deposit or withdrawal at month k.
III: **(wrong)** The account balance at month k.
IV: **(wrong)** The total interest earned at month k.
V: **(wrong)** I do not know.

Solution 1:

In the bank account model, $u[k]$ represents the monthly deposit or withdrawal at month k. The state $x[k]$ represents the account balance, and $r[k]$ is the interest rate.

notes

- see the associated solution(s), if compiled with that ones :)

Question 6

In the Lotka-Volterra model, what does the term $\beta y_{\text{prey}} y_{\text{pred}}$ represent?

Potential answers:

I: **(wrong)** The natural growth rate of the prey population.
II: **(wrong)** The natural death rate of the predator population.
III: **(correct)** The interaction between prey and predator populations.
IV: **(wrong)** The external disturbance affecting the system.
V: **(wrong)** I do not know.

Solution 1:

In the Lotka-Volterra model, the term $\beta y_{\text{prey}} y_{\text{pred}}$ represents the interaction between the prey and predator populations, where β is the predation rate.

notes

- see the associated solution(s), if compiled with that ones :)

Question 7

In the exponential growth model $x^+ = \alpha x + \beta u$, what does α represent?

Potential answers:

I: **(correct)** The growth rate of the system.
II: **(wrong)** The input to the system.
III: **(wrong)** The output of the system.
IV: **(wrong)** The disturbance affecting the system.
V: **(wrong)** I do not know.

Solution 1:

In the exponential growth model $x^+ = \alpha x + \beta u$, α represents the growth rate of the system, determining how the state x evolves over time.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module “state space representations”

- a set of variables is a state vector if it satisfies for that model the separation principle, i.e., the current state vector “decouples” the past with the future
- state space models are finite, and first order vectorial models

notes

- the most important remarks from this sub-module are these ones