Table of Contents I

notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

what is the superposition principle, and what does it imply

notes

-

Contents map

developed content units	taxonomy levels
superposition principle	u1, e1

prerequisite content units	taxonomy levels
LTI RR	u1, e1

notes

-

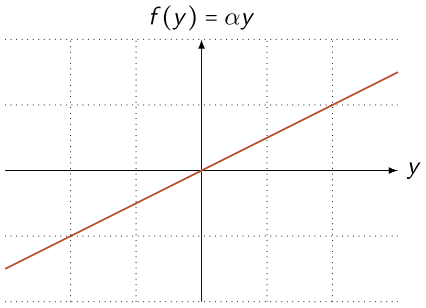
Main ILO of sub-module “what is the superposition principle, and what does it imply”

Describe the importance of the superposition principle to analyze LTI systems

notes

- by the end of this module you shall be able to do this

Starting with graphs



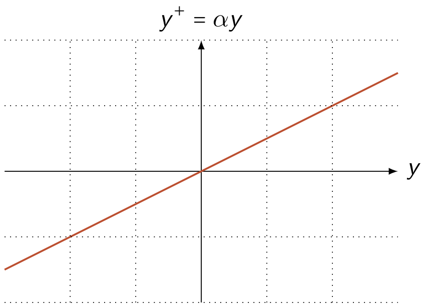
implications/definition of linearity:

- $f(x + y) = f(x) + f(y)$
- $f(\alpha y) = \alpha f(y)$

notes

- looking at this graph, we note that these two properties hold
- and this holds only because of linearity. If we are having an affine map, for example, the second would not hold

What if we interpret this as a RR?



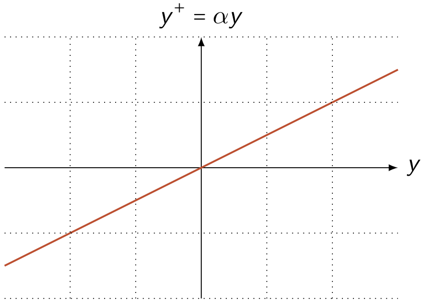
$\Longrightarrow$ an LTI system, for which

$$y^+ = \alpha y \quad \text{is solved by} \quad y[k] = y[0]\alpha^k \quad \forall y[0], \alpha, k$$

notes

- you may verify this solution by doing the direct verification

And can we build on top of this?



$\implies$ an LTI system, for which

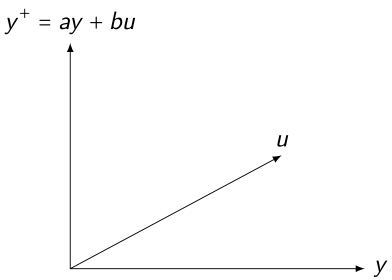
- $y'[0] = 2 \mapsto y'[k] = 2\alpha^k$
- $y''[0] = 3 \mapsto y''[k] = 3\alpha^k$
- $y'''[0] = 3 + 2 \mapsto y'''[k] = (3 + 2)\alpha^k$

$$y'[0] + y''[0] \quad \mapsto \quad y'[k] + y''[k]$$

notes

- looking at what we found in the previous module, this holds because the solutions to linear RRs are powers passing by the initial conditions and whose base is always α
- and this holds only because of linearity

Further generalization



- $\{y'[0], u'\} \mapsto y'[k]$
- $\{y''[0], u''\} \mapsto y''[k]$
- $\{y'[0] + y''[0], u' + u''\} \mapsto y'[k] + y''[k]$

notes

- we can then generalize the previous result in this way, when we have linearity

Aiding intuitions with math

Linearity implies that if $\{y', u', y'[0]\}$ and $\{y'', u'', y''[0]\}$ satisfy

$$\begin{cases} y'[k+1] &= ay'[k] + bu'[k] \\ y'[0] &= y'_0 \\ \\ y''[k+1] &= ay''[k] + bu''[k] \\ y''[0] &= y''_0 \end{cases} \tag{1}$$

then their sum also satisfies

$$\begin{cases} \left(\alpha' y'[k+1] + \alpha'' y''[k+1]\right) &= a\left(\alpha' y'[k] + \alpha'' y''[k]\right) + b\left(\alpha' u'[k] + \alpha'' u''[k]\right) \\ \alpha' y'[0] + \alpha'' y''[0] &= \alpha' y'_0 + \alpha'' y''_0 \end{cases} \tag{2}$$

notes

- let's start realizing that this holds because derivatives hold

Rephrasing

Linearity implies that if $\{y', u', y'[0]\}$ and $\{y'', u'', y''[0]\}$ satisfy the RR then also their sum $\{y' + y'', u' + u'', y'[0] + y''[0]\}$ satisfies the RR.

The superposition principle in words

in LTI systems
combining inputs and initial conditions
produces a total effect
that is the linear combination
of that effects
one would get with the individual causes
each acting separately

notes

- this is the same thing in the previous slide, written in words
- then the previous math basically says this

Important: the superposition principle works with any LTI

Will be repeated and stated again precisely later on

the proof holds for every system that generalizes $y^{+} = ay + bu$,
i.e., every "linear combination of temporal shifts of y = linear combination of temporal shifts of u"

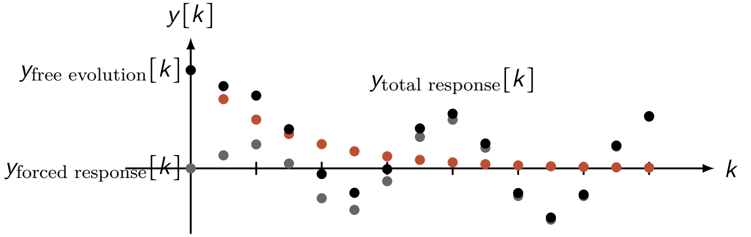
notes

- we have moreover this generalization, that we will see again and again, since derivatives are linear

Superposition principle $\Longrightarrow$ response of LTIs = free evolution + forced response

assume:

- $y^{+} = ay + bu$
- $\{u[k] = 0[k], \quad y[0] \neq 0\}$ causes $y_{\text{free evolution}}[k]$
- $\{u[k] \neq 0[k], \quad y[0] = 0\}$ causes $y_{\text{forced response}}[k]$



then $\{u[k] \neq 0[k], y[0] \neq 0\}$ causes $y_{\text{free evolution}}[k] + y_{\text{forced response}}[k]$

notes

- the generalization in the previous slide can immediately be useful to show something very important, i.e., that we can decompose the general solution in to main parts, as here

A mnemonic scheme

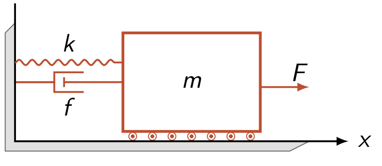
(only for LTI systems!!)

$$(u, y_0) = (0, y_0) + (u, 0)$$

total response = free evolution + forced response

notes

Continuing with some intuitions



Discussion: how will the cart move if I use $u[k] = \sin(\omega k T)$ starting from a resting state? (only intuitively, assuming everything ideal)
And what about if $u[k] = 2\sin(\omega k T)$?
And what about $u[k] = \sin\left(\omega' k T\right) + \sin\left(\omega'' k T\right)$?
And what about $u[k] = \alpha' \sin\left(\omega' k T\right) + \alpha'' \sin\left(\omega'' k T\right)$?

notes

- we will see later on precisely; for now let's say that it moves somehow
- here intuition may not help: anyway it makes the same movement as before but with twice the amplitude
- here intuition may again not help: it makes the sum of the two movements
- here intuition may again not help: it makes the sum of the two movements scaled

Summarizing

Describe the importance of the superposition principle to analyze LTI systems

- it makes us able to say "total $=$ free $+$ forced"

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

-

Suggestion

part of the SciPy library (`scipy.signal`) provides tools for working with LTI systems, including creating transfer functions, state-space representations, and analyzing system responses (stuff that will be seen in the next modules)

notes

- check this library, you will use it

Self-assessment material

notes

Question 1

What is the primary implication of the superposition principle in LTI systems?

Potential answers:

- I: **(correct)** The total response is the sum of the free evolution and the forced response.
- II: **(wrong)** The system response is always exponential.
- III: **(wrong)** The system response is independent of the initial conditions.
- IV: **(wrong)** The system response is nonlinear.
- V: **(wrong)** I do not know

Solution 1:

The superposition principle in LTI systems allows us to decompose the total response into two components: the free evolution (response due to initial conditions) and the forced response (response due to external inputs). This is a fundamental property of LTI systems.

What is the superposition principle and what does it imply 2

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

Which of the following properties is essential for a system to be considered linear?

Potential answers:

- I: **(wrong)** The system response is always sinusoidal.
- II: **(correct)** The system satisfies the properties $f(x+y) = f(x) + f(y)$ and $f(\alpha y) = \alpha f(y)$.
- III: **(wrong)** The system response is independent of the input.
- IV: **(wrong)** The system response is always zero for zero input.
- V: **(wrong)** I do not know

Solution 1:

For a system to be linear, it must satisfy the properties of additivity ($f(x+y) = f(x) + f(y)$) and homogeneity ($f(\alpha y) = \alpha f(y)$). These properties are fundamental to the definition of linearity.

What is the superposition principle and what does it imply 3

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

What happens to the response of an LTI system if the input is scaled by a factor α?

Potential answers:

I: **(wrong)** The response becomes nonlinear.
II: **(wrong)** The response remains unchanged.
III: **(correct)** The response is scaled by the same factor α.
IV: **(wrong)** The response becomes zero.
V: **(wrong)** I do not know

Solution 1:

In an LTI system, scaling the input by a factor α results in the response being scaled by the same factor α. This is a direct consequence of the homogeneity property of linear systems.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

What is the significance of the superposition principle in analyzing LTI systems?

Potential answers:

I: **(wrong)** It allows us to ignore the initial conditions.
II: **(correct)** It allows us to decompose the system response into free evolution and forced response.
III: **(wrong)** It makes the system response independent of the input.
IV: **(wrong)** It ensures the system response is always exponential.
V: **(wrong)** I do not know

Solution 1:

The superposition principle is crucial in LTI systems because it allows us to break down the total response into two parts: the free evolution (due to initial conditions) and the forced response (due to external inputs). This decomposition simplifies the analysis of complex systems.

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

Which of the following statements is true about the superposition principle in LTI systems?

Potential answers:

I: **(wrong)** It only applies to nonlinear systems.
II: **(wrong)** It is only valid for zero initial conditions.
III: **(correct)** It states that the response to a sum of inputs is the sum of the responses to each input individually.
IV: **(wrong)** It implies that the system response is always sinusoidal.
V: **(wrong)** I do not know

Solution 1:

The superposition principle in LTI systems states that the response to a sum of inputs is the sum of the responses to each input individually. This is a direct consequence of the linearity property of LTI systems.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module "what is the superposition principle, and what does it imply"

- superposition principle helps logically separating specific causes into specific effects
- linear RRs $\Longrightarrow$ superposition principle
- superposition principle $\Longrightarrow$ "whole = free + forced"
- nonlinear systems WON'T satisfy this principle!

notes

- the most important remarks from this sub-module are these ones