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notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

building and interpreting phase portraits for RRs

notes

Contents map

developed content units	**taxonomy levels**
phase portrait	u1, e1

prerequisite content units	**taxonomy levels**
ODE	u1, e1
RR	u1, e1

notes

Main ILO of sub-module "building and interpreting phase portraits for RRs"

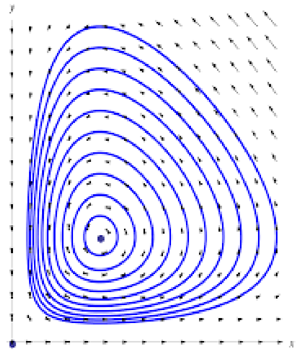
Construct and interpret phase portraits of first- and second-order autonomous RRs using qualitative analysis techniques

notes

- by the end of this module you shall be able to do this

Starting with a CT example: a Lotka-Volterra model:

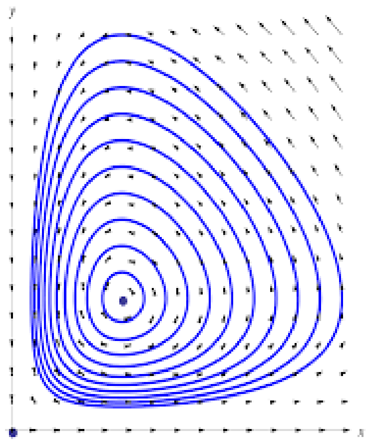
$$\begin{cases} \dot{y}_{\text{rabbits}} &= 0.4 \cdot y_{\text{rabbits}} - 0.5 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \\ \dot{y}_{\text{foxes}} &= -3 \cdot y_{\text{foxes}} + 0.7 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \end{cases}$$



notes

- if we were doing things graphically by hand we would eventually get this

Phase Portrait in CT = a graphical representation of the trajectories of a dynamical system in the state space



notes

- A phase portrait provides insight into the qualitative behavior of a system without requiring explicit solutions.
- It helps visualize equilibria, stability, and the general flow of solutions in state space.

But what happens if we discretize the system?

$$\begin{cases} \dot{y}_{\text{rabbits}} &= 0.4 \cdot y_{\text{rabbits}} - 0.5 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \\ \dot{y}_{\text{foxes}} &= -3 \cdot y_{\text{foxes}} + 0.7 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \end{cases}$$

$$\Downarrow$$

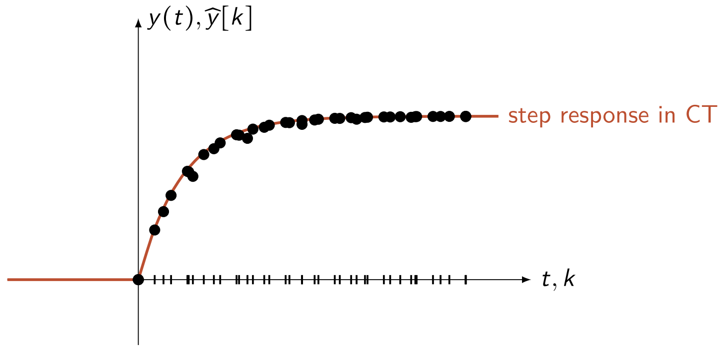
$$\begin{cases} y^{+}_{\text{rabbits}} &= 1.3 \cdot y_{\text{rabbits}} - 0.7 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \\ y^{+}_{\text{foxes}} &= 0.8 \cdot y_{\text{foxes}} + 0.9 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \end{cases}$$

changing discretization time T will make the parameters change, and thus get different approximations

notes

- the fact is that we get an approximated version of the system, and we may have different discretizations and thus approximation levels

Important concept: as T decreases, DT discretizations become more and more accurate representations of the CT dynamics



notes

- As T decreases, the discrete-time solution (dashed lines) converges to the continuous-time solution (solid blue line).
- This demonstrates the accuracy of Euler's backward method for smaller time steps.

Summarizing

Construct and interpret phase portraits of first- and second-order autonomous RRs using qualitative analysis techniques

- the plots are basically the same that one may get from the phase portraits relative to ODEs, but they are discrete versions of them
- the accuracy of the discretization depends on the sampling period, and thus one shall always be a bit wary of the results one get

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

Tutorial on how to plot phase portraits

`https://aleksandarhaber.com/phase-portraits-of-state-space-models-and-differential-equations-in-python/`

notes

- that tutorial is very well made, check it

Self-assessment material

notes

-

Question 1

What is the primary purpose of a phase portrait for a discrete-time system?

Potential answers:

I: **(wrong)** To compute the exact solution of the system
II: **(correct)** To visualize the qualitative behavior of the system's trajectories in state space
III: **(wrong)** To determine the numerical stability of the system
IV: **(wrong)** To solve the system's differential equations analytically
V: **(wrong)** I do not know

Solution 1:

The primary purpose of a phase portrait is to visualize the qualitative behavior of the system's trajectories in state space, including equilibria, stability, and general flow patterns.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

What happens to the accuracy of a discrete-time system's phase portrait as the discretization time T decreases?

Potential answers:

I: **(wrong)** The phase portrait becomes less accurate
II: **(correct)** The phase portrait becomes more accurate, converging to the continuous-time solution
III: **(wrong)** The phase portrait remains unchanged
IV: **(wrong)** The phase portrait becomes unstable
V: **(wrong)** I do not know

Solution 1:

As T decreases, the discrete-time system's phase portrait becomes more accurate, converging to the continuous-time solution.

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

When discretizing the Lotka-Volterra model, what is the effect of changing the discretization time T?

Potential answers:

I: **(wrong)** The system's equilibria change
II: **(correct)** The parameters of the discretized system change, leading to different approximations
III: **(wrong)** The system becomes unstable
IV: **(wrong)** The system's trajectories become chaotic
V: **(wrong)** I do not know

Solution 1:

Changing the discretization time T alters the parameters of the discretized system, resulting in different approximations of the continuous-time dynamics.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

What does a stable equilibrium point in a discrete-time phase portrait indicate?

Potential answers:

I: **(wrong)** Trajectories diverge away from the equilibrium point
II: **(correct)** Trajectories converge to the equilibrium point over time
III: **(wrong)** The system exhibits periodic behavior
IV: **(wrong)** The system becomes chaotic
V: **(wrong)** I do not know

Solution 1:

A stable equilibrium point in a discrete-time phase portrait indicates that trajectories converge to the equilibrium point over time.

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

What does a closed loop in a discrete-time phase portrait typically represent?

Potential answers:

- I: **(wrong)** A stable equilibrium point
- II: **(wrong)** An unstable equilibrium point
- III: **(correct)** Periodic or quasi-periodic behavior
- IV: **(wrong)** Chaotic behavior
- V: **(wrong)** I do not know

Solution 1:

A closed loop in a discrete-time phase portrait typically represents periodic or quasi-periodic behavior, where the system's state repeats over time.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module "building and interpreting phase portraits for RRs"

- A phase portrait is a graphical representation of a dynamical systems trajectories in state space.
- Phase portraits provide qualitative insight into system behavior without requiring explicit solutions.
- First-order systems have a one-dimensional state space, while second-order systems require two dimensions, etc.
- The smaller the sampling period T, the closer the discrete-time phase portraits is to the one one would get from the continuous time version of the system

notes

- the most important remarks from this sub-module are these ones