Table of Contents I

notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

compute the equilibria of the system

notes

Contents map

developed content units	**taxonomy levels**
equilibrium	u1, e1

prerequisite content units	**taxonomy levels**
RR	u1, e1

notes

Main ILO of sub-module "compute the equilibria of the system"

Compute the equilibria of a RR by solving for stationary points

notes

- by the end of this module you shall be able to do this

Is this in equilibrium?

notes

- yes, intuitively

Are these in equilibrium, while falling?

notes

- no, intuitively

Equilibrium = a trajectory that is constant in time

$$y[k] = \text{constant}$$

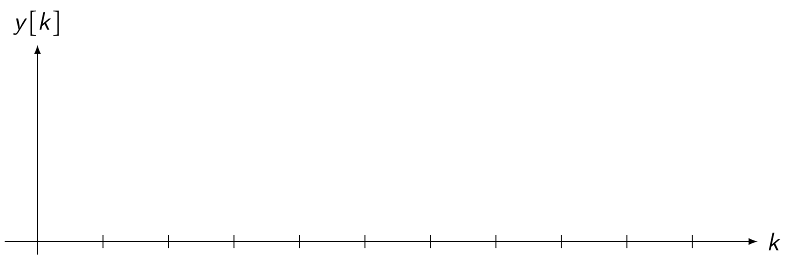
notes

- the simplest trajectories are that ones that seem simple constant numbers

Example

average temperature of a yogurt taken out from the fridge into a very large room whose temperature is 20 degrees, forward-Euler discretized with a sampling time $T = 1$ seconds:

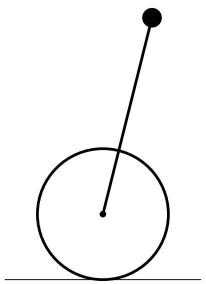
$$\dot{y} = -0.5\big(y - 20\big) \quad \mapsto \quad y^{+} = 0.5y + 10$$



notes

- let's do a practical example of a simple system
- here we can see that starting from any initial temperature we ideally tend to go that of the room
- and if we start already at the temperature of the room we have an equilibrium

What does it mean that this system is in equilibrium from an intuitive point of view?



notes

- let's start with the simplest concept, the one of equilibrium, through a practical example: a segway is in equilibrium if it is perfectly upright or if it is laying on the floor
- the term equilibrium means, from an intuitive point of view, 'things do not move', and thus the trajectories stay constant

What does it mean that this system is in equilibrium from a mathematical point of view?

equilibrium means $\boldsymbol{y}^{+} = \boldsymbol{y}$

this implies

$$\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}} \text{ is an equilibrium point iff } \boldsymbol{y}^{+} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right)$$

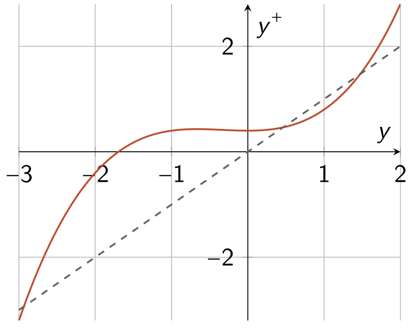
i.e., the equilibria of a system are the zeros of $\boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right)$

notes

- 'things do not move' then mathematically translate into $\dot{\boldsymbol{y}} = \boldsymbol{0}$
- $\dot{\boldsymbol{y}} = \boldsymbol{0}$ means that we are looking for that points that make $\boldsymbol{f}$ zero
- note that "iff" means if and only if

Equilibra as the zeros of $y^+ = f(y)$, graphically

Exemplified situation of *autonomous* single output systems:

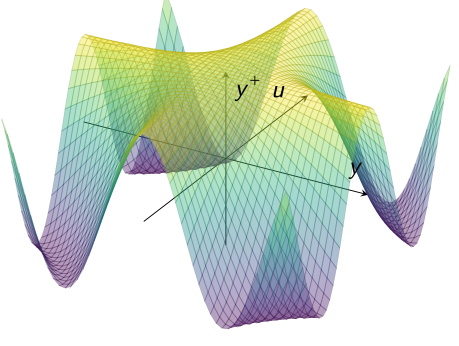


notes

- so one can see the equilibria from the crossings of $\boldsymbol{f}$ with the 'x axis' (note that this is a scalar case, and the "autonomous" case where there is no input u, so a simplified case)

Equilibra as the zeros of $\boldsymbol{f}$, graphically

Exemplified situation of SISO (single input single output) systems:



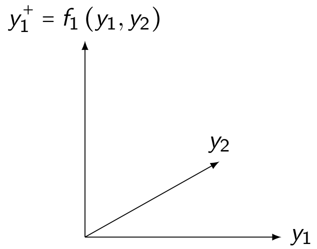
https://www.geogebra.org/classic/mmppe6hs

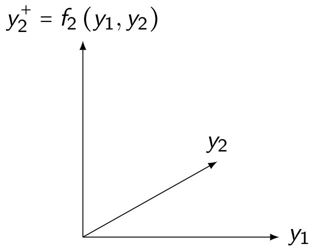
notes

- one can see the equilibria from the crossings of $\boldsymbol{f}$ with the bisector 'yu plane' in case we have more dimensions

Equilibra as the zeros of $\boldsymbol{f}$, graphically

Exemplified situation of automonous multiple output systems:





(remember: $\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}$ equilibrium iff $\boldsymbol{f}(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) = \boldsymbol{y}_{\text{eq}}$, i.e., all the components simultaneously!)

notes

- note that if one has more than one y then it must be $y_i^+ = 0$ for all the i's simultaneously at the same point

equilibria = extremely important topic in automatic control!

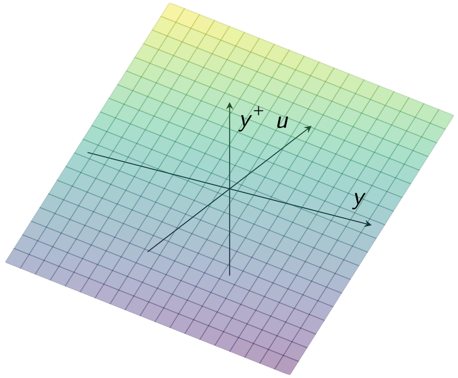
will be analysed in more details later on in this course
and much more extensively in others
(feat. Lyapunov, Krasovskii, La-Salle among others)

notes

- this module is on ODEs and not on equilibria; however better clarifying that this is an extremely important thing

Discussion: what are the equilibria in this case?

$y^+ = ay + bu$



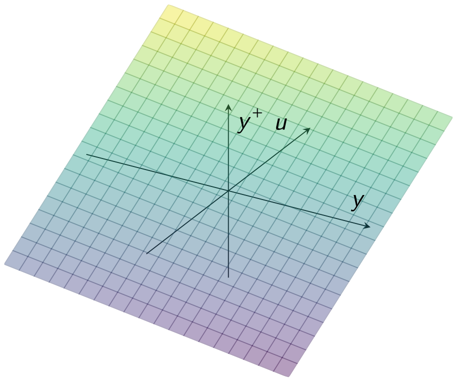
$(y_0 = 0, u = 0)$ is always an equilibrium for these systems!

notes

-
- $y^+ = y$, so if $a = 0$ then any y but it has to be $u = 0$, if $a \neq 0$ then it has to be so that $ay + bu = 0$
- it is also interesting to note this

Discussion: can we have for this specific RR an equilibrium if $u \neq \text{constant}$?

$y^+ = ay + bu$

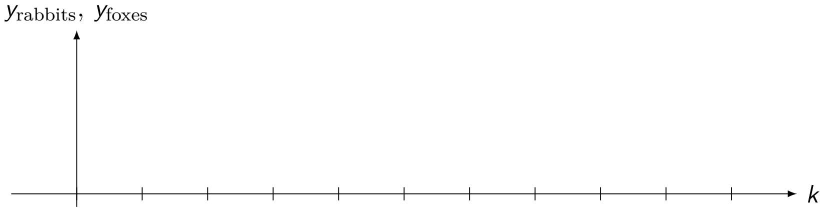


notes

- no

Another example: a Lotka-Volterra model (≠ real world):

$$\begin{cases} y^+_{\text{rabbits}} = 1.4 \cdot y_{\text{rabbits}} - 0.5 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \\ y^+_{\text{foxes}} = 0.7 \cdot y_{\text{foxes}} + 0.7 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \end{cases}$$



notes

- this you should be immediately able to find an equilibrium, that is 0,0. The other one there is a bit of algebra to do but it comes natually

Summarizing

Compute the equilibria of a RR by solving for stationary points

- put $y = f(y,u) \implies y - f(y,u) = 0$, and compute the corresponding points. It may be that there is the need to put all the various inputs $u = \text{constant}$ (or disturbances $d = \text{constant}$)

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

Root finding in python

https://pythonnumericalmethods.studentorg.berkeley.edu/notebooks/chapter19.05-Root-Finding-in-Python.html

notes

- this page explains how to find the zeros of any function, and thus also of $y - f(y, u) = 0$

Self-assessment material

notes

Question 1

Which of the following best defines an equilibrium point in a dynamical system?

Potential answers:

- I: **(wrong)** A point where the system's state constantly increases.
- II: **(correct)** A point where the system's state remains unchanged over time.
- III: **(wrong)** A point where the system's state oscillates periodically.
- IV: **(wrong)** A point where the system's state diverges exponentially.
- V: **(wrong)** I do not know

Solution 1:

An equilibrium point is defined as a state where the system remains unchanged over time, meaning $y^+ = y$.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

How can equilibrium points be identified in a graphical representation of $y^+ = f(y)$?

Potential answers:

I: **(wrong)** At points where $f(y)$ reaches its maximum value.
II: **(wrong)** At points where $f(y)$ crosses the y-axis.
III: **(wrong)** At points where $f(y)$ is strictly increasing.
IV: **(correct)** At points where $f(y) = y$, indicating that the system remains unchanged.
V: **(wrong)** I do not know

Solution 1:

Equilibrium points are found where $y^+ = f(y)$ intersects the identity line $y^+ = y$, ensuring that the state remains constant.

- compute the equilibria of the system 3

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

Which of the following systems is most likely to be in equilibrium?

Potential answers:

I: **(wrong)** A ball rolling down a hill.
II: **(correct)** A balancing robot standing perfectly upright and not moving.
III: **(wrong)** A pendulum swinging back and forth.
IV: **(wrong)** A falling hailstone.
V: **(wrong)** I do not know

Solution 1:

An equilibrium state occurs when there is no change in the systems state over time, such as when a balancing robot remains perfectly upright without movement.

- compute the equilibria of the system 4

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

For the discrete-time system $y^+ = 0.5y + 10$, what is the equilibrium value of y?

Potential answers:

I: **(wrong)** 0
II: **(wrong)** 10
III: **(correct)** 20
IV: **(wrong)** 40
V: **(wrong)** I do not know

Solution 1:

To find equilibrium, solve $y^+ = y$:
$y = 0.5y + 10$
$y - 0.5y = 10$
$0.5y = 10$
$y = 20$

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

In a linear time-invariant system, an equilibrium point can be computed by:

Potential answers:

I: **(correct)** Setting the system dynamics to zero and solving for state and input values.
II: **(wrong)** Taking the time derivative of the system matrix.
III: **(wrong)** Finding the eigenvalues of the system matrix.
IV: **(wrong)** Taking the integral of the system dynamics.
V: **(wrong)** I do not know

Solution 1:

Equilibrium is found by setting $y^+ = y$ in the system equation and solving for the corresponding state and input values.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module "compute the equilibria of the system"

- Equilibria in dynamical discrete time systems correspond to points where the system's state does not change over time.
- Autonomous time-varying RRs can have equilibria, but their location may vary with time.
- Some dynamical systems may not have equilibria, particularly if they involve unbounded growth.
- Non-autonomous LTI RRs can have equilibria only if the input $u(t)$ remains constant over time.

notes

- the most important remarks from this sub-module are these ones