Table of Contents I

notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

Is this time series a solution of this recurrence relation?

notes

Contents map

developed content units	taxonomy levels
recurrence relation	u1, e1

prerequisite content units	taxonomy levels
delay operator	u1, e1

notes

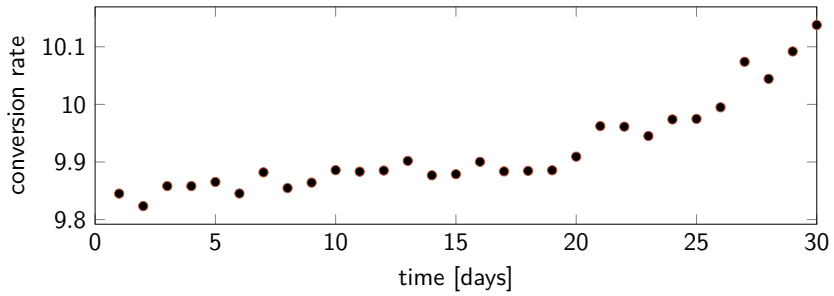
Main ILO of sub-module
"Is this time series a solution of this recurrence relation?"

Decide whether a given time series is a solution to a specified recurrence relation by direct verification

notes

- by the end of this module you shall be able to do this

What is a time series?

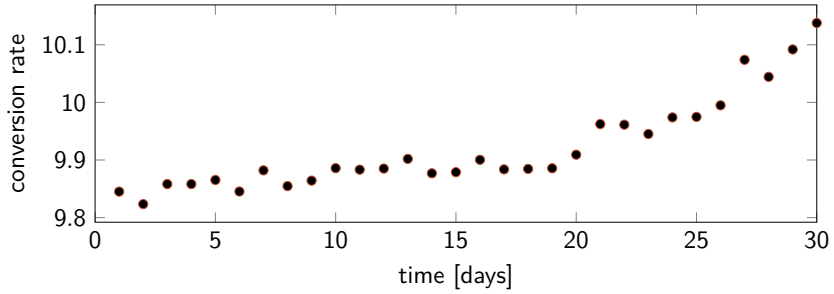


$y(kT)$, or $y[k]$ for simplicity of notation (assuming time discrete in this module)

notes

- a signal is any time-dependent (or space-dependent) variation of a physical quantity that conveys information. In engineering and science, signals can take many forms, such as electrical voltages, sound waves, images, or sequences of numbers in a digital system. A time series is then a discrete-time version of a signal

What is the equivalent of a derivative for discrete-time signals?

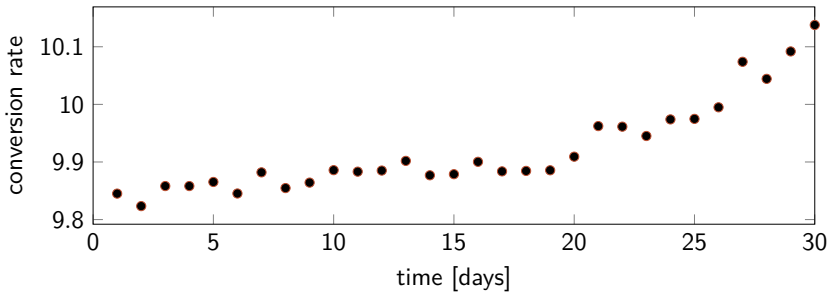


$$y[k] \quad \mapsto \quad \frac{y[k]-y[k-1]}{T} = \frac{y[k]-y[k]q^{-1}}{T} = y[k]\left(\frac{1-q^{-1}}{T}\right)$$

notes

- we will extensively use this notation in the course; basically always use the operator q to indicate time shifts

Would you say that $y[k] = y[k]q^{-1}$, in this case?



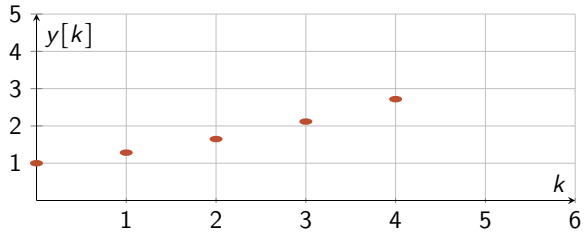
notes

- definitely not; the signal is not constant at all

"uhm, where are we going with all this stuff?"

↦ be able to do forecasts

would you be able to compute $y[6]$ from this graph, if you knew that $y[k] = 1.2y[k+1]$?



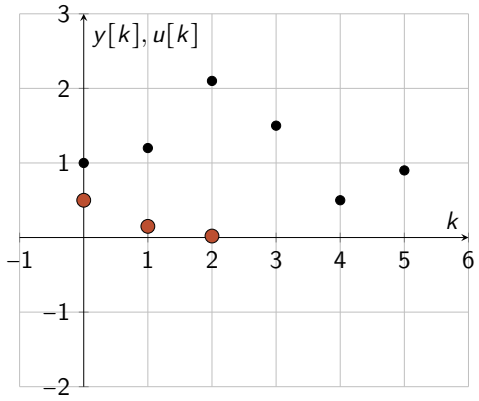
notes

- we will see how a RR (Ordinary Differential Equation) describes how a system's state evolves over time based on its current state and inputs. By the end of the course it will be obvious how solving the RR with given initial conditions, we can predict future states. This is used in control to enable forecasting, that is essentially by propagating system dynamics forward in time. We will also see though that the accuracy of forecasts depends on how well the RR models the real-world system (something that may be more or less good)

a RR is a tool to produce forecasts

Example: $y[k+1] = 0.5y^2[k]u[k]$

k	$y[k]$	$u[k]$
0	0.5	1
1	0.15	1.2
2	≈0.02	2.1
3	⋯	1.5
4	⋯	0.5
5	⋯	0.9



notes

- so this is an important message
- watch out at the notation; we will always mean signals, and not scalars

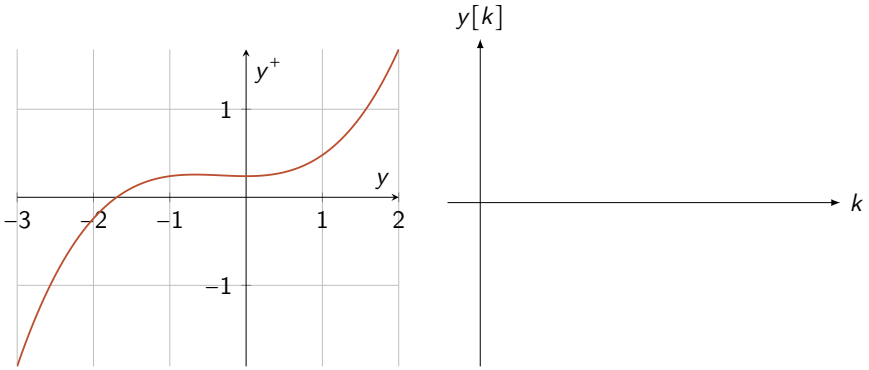
Notation

symbol	↦	meaning	alternative 1	alternative 2
⋮		⋮	⋮	⋮
y^{+++}		$y[k+3]$	yq^3	$y^{[3]}$
y^{++}		$y[k+2]$	yq^2	$y^{[2]}$
y^{+}		$y[k+1]$	yq^1	$y^{[1]}$
y		$y[k]$	yq^0	$y^{[0]}$
y^{-}		$y[k-1]$	yq^{-1}	$y^{[-1]}$
y^{--}		$y[k-2]$	yq^{-2}	$y^{[-2]}$
y^{---}		$y[k-3]$	yq^{-3}	$y^{[-3]}$
⋮		⋮	⋮	⋮

notes

- watch out at the notation; we will always mean signals, and not scalars

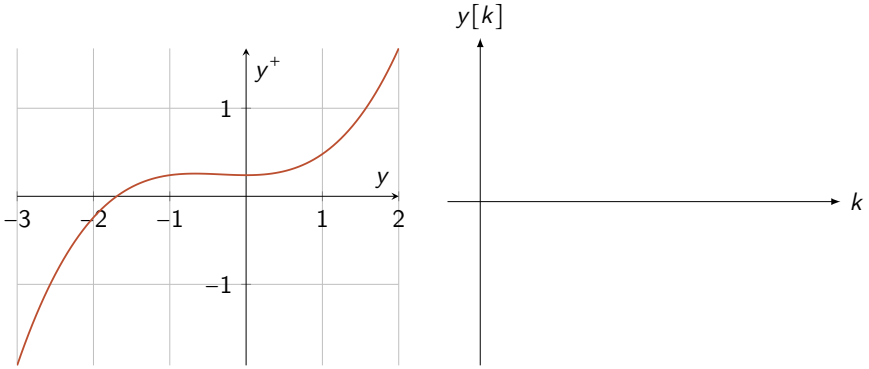
But what does it mean to solve a RR, graphically?



notes

- before continuing we need though to reflect on what it means to solve a differential equation. Let's take this example to investigate this concept
- remember that all the symbols here actually mean signals in time, i.e., they should be intended as $y^+ = y + u$ ideally with t in a certain domain (typically $[0, +\infty)$)
- for example this combination of signals does not satisfy the equation

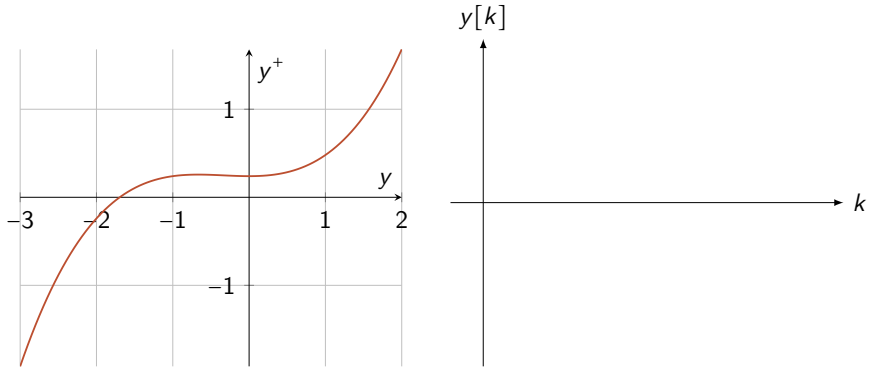
Is knowing the RR enough to be able to generate a trajectory?



notes

- no, there is the need for defining an initial condition to be able to draw the trajectory

Does $\{y[k] = \cos[k], y[0] = 1\}$ solve this RR?



notes

- no, because that specific signal does not satisfy the RR

Are we done with this?

Decide whether a given function is a solution to a specified RR by direct verification

→ no, there are still a lot of cases we shall cover

notes

- by the end of this module you shall be able to do this

Notation time!

In control, modelling a dynamical system = defining

$$\boldsymbol{y}^+ = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\theta}),$$

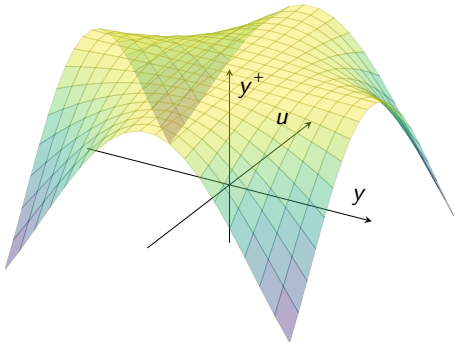
thus defining:

- the variables
  - $\boldsymbol{u}$ = inputs *(i.e., what we can steer)*
  - $\boldsymbol{d}$ = disturbances *(i.e., what we cannot steer but that still influences the system)*
  - $\boldsymbol{y}$ = outputs *(i.e., what we are interested into)*
- the shape of $\boldsymbol{f}$
- the value of its parameters $\boldsymbol{\theta}$
- bold font = vector

notes

- let's now generalize the RR before to something that can be applied to more cases
- let's do this definition, where the names are given in this way for historical reasons
- for example the Lotka Volterra model that we will see below is a specific example of this way of writing things, where there is no u and d by the way
- once again $\boldsymbol{f}$ has the meaning of indicating "where" the system is going towards, in time - also this will be more clear soon

A graphical example of $y^+ = f(y, u)$

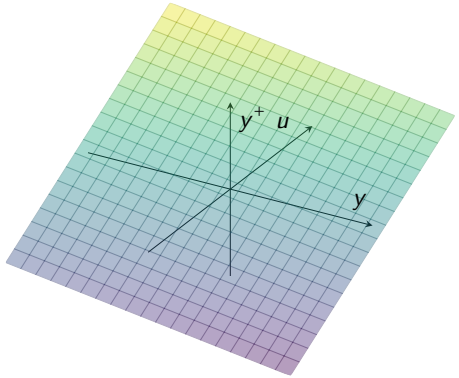


https://www.geogebra.org/classic/mmppe6hs

notes

- the simplest trajectories are that ones that seem simple constant numbers

The special RR $y^+ = ay + bu$



notes

- you should have already seen these RRs (or be able to derive them) from physics and electronics

Watch out: converting an ODE to a RR is not just substituting 'dots' with 'pluses'

temperature of the center of a cake in an oven whose temperature is 200 degrees:

$$\dot{T} = -0.5\big(T - 200\big) \quad ? \mapsto ? \quad T^+ = -0.5\big(T - 200\big)$$

notes

- let's do a practical example of a very simple RR
- here we can see that starting from any T we ideally tend to go to T_a
- but how? Let's see

notes

- https://en.wikipedia.org/wiki/The_Treachery_of_Images

Important point: model ≠ real world

Ceci n'est pas un gâteau.

$$T^+ = f(T)$$

notes

- we will re-consider this point later on, but for now it has to be obvious that this is an approximated description of reality
- why we use them and what are the usages we'll do about this, we'll see later on in the course

Summarizing

Decide whether a given function is a solution to a specified RR by direct verification

- check y, compute $f(y)$, compute y^+
- does $f(y) = y^+$?
- same apply for higher orders / more complex RRS from notational perspectives

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

Solving RRs

`https://www.geeksforgeeks.org/recurrence-relation-in-python/`

notes

- check that page

Self-assessment material

notes

Question 1

Which of the following best describes what it means for a function $y[k]$ to be a solution of a RR?

Potential answers:

I: **(wrong)** It satisfies the RR for at least one value of k.
II: **(correct)** It satisfies the RR for all values of k in its domain.
III: **(wrong)** It approximately satisfies the RR within a certain error margin.
IV: **(wrong)** It satisfies the RR only at integer values of k.
V: **(wrong)** I do not know

Solution 1:

A function is a solution of a RR if it satisfies the equation for all values of k within its domain. A solution must be valid throughout the considered interval, not just at isolated points.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

What additional information is needed to uniquely determine a solution of a RR?

Potential answers:

I: **(wrong)** The function $y[k]$ itself.
II: **(correct)** An initial condition specifying the value of y at a given time.
III: **(wrong)** A boundary condition at two different points.
IV: **(wrong)** The highest-order difference of y.
V: **(wrong)** I do not know

Solution 1:

An initial condition provides the necessary information to select a unique solution from the family of possible solutions to a differential equation. Without it, multiple solutions may exist.

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

Given the RR $y^+ = y$, which of the following functions is a solution?

Potential answers:

I: **(wrong)** $y[k] = 0$
II: **(correct)** $y[k] = C$, where C is a constant.
III: **(wrong)** $y[k] = \sin k$
IV: **(wrong)** $y[k] = \frac{1}{k+1}$
V: **(wrong)** I do not know

Solution 1:

The function $y[k] = C$ satisfies the equation since it matches the right-hand side of the RR.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module

"Is this time series a solution of this recurrence relation?"

- a function is a solution of a RR if it satisfies the equation for all values in its domain
- initial conditions are necessary to uniquely determine a solution

notes

- the most important remarks from this sub-module are these ones