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notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

explain what BIBO stability means

notes

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Contents map

developed content units	taxonomy levels
unbounded signal	u1, e1
bounded signal	u1, e1
BIBO stability	u1, e1

prerequisite content units	taxonomy levels
ODE	u1, e1

notes

▪

Main ILO of sub-module "explain what BIBO stability means"

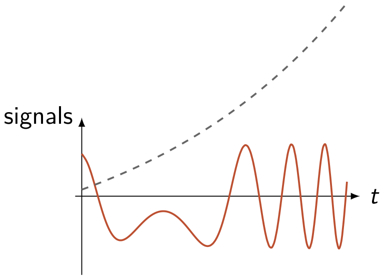
Graphically explain the definition of BIBO (Bounded-Input Bounded-Output) stability and its connection to system behavior

Give examples of systems that are BIBO stable or not, and motivate their properties with physical intuitions

notes

- by the end of this module you shall be able to do this

Definitions: bounded and unbounded signals

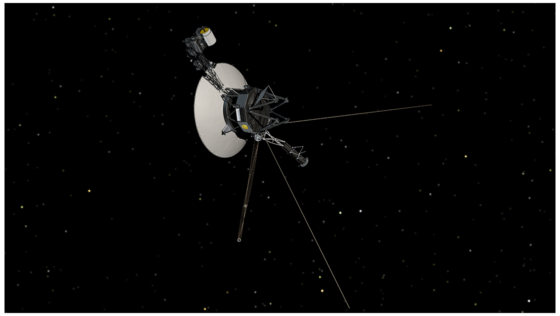


bounded: I can find an M for which $|y(t)| < M$ for every t. **Unbounded** otherwise

notes

- by the end of this module you shall be able to do this

Intuition



simplified model = single integrator: $\dot{v} = bu$

$$u(t) = \text{const.} \neq 0 \quad \Longrightarrow \quad \lim_{t \to +\infty} v(t) = ?$$

notes

- if you make the spacecraft keep accelerating, its velocity will become that of light
- this is an example where even if we have a bounded input, we get an unbounded output

Interesting difference

$\dot{v} = 0v + bu \implies$ constant non null $u(t)$ causes $v(t)$ unbounded

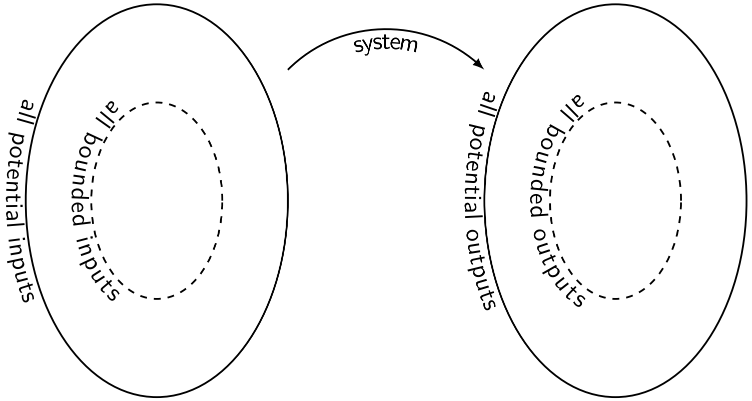
$\dot{v} = av + bu, a < 0 \implies$ constant non null $u(t)$ causes $v(t)$ bounded

notes

- so let's consider a simple single integrator. It is obvious that $v(t)$ is going to be the integral of $bu(t)$, so the integral of a constant signal is the area formed that a sort of rectangle with an infinitely long basis, i.e., infinite, thus $v(t)$ diverges
- at the same time if we have a first order system, this does not happen
- indeed the corresponding impulse response is be^{at}, and thus (one can verify this immediately also from the definition of convolution) $y(t)$ will tend to $\overline{u}$ times the integral of be^{at} between 0 and $+\infty$

Visualizing this with avocados

for $\dot{v} = 0v + bu$ some bounded $u(t)$'s lead to unbounded $v(t)$'s

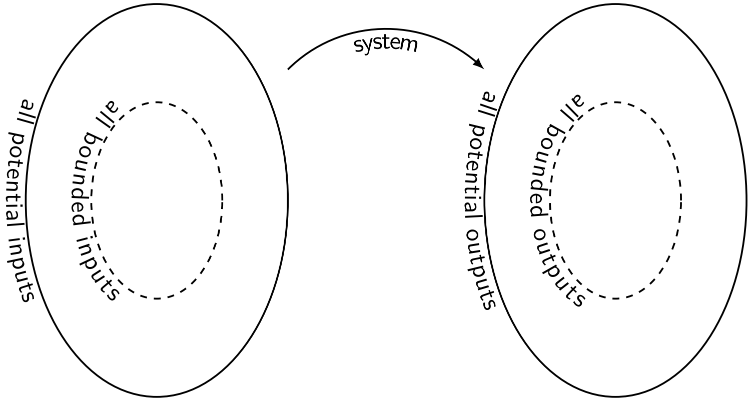


notes

- we now see that some inputs get mapped into unbounded forced responses

Visualizing this with avocados

for $\dot{v} = 0v + bu$ not all the bounded $u(t)$'s lead to unbounded $v(t)$'s, though

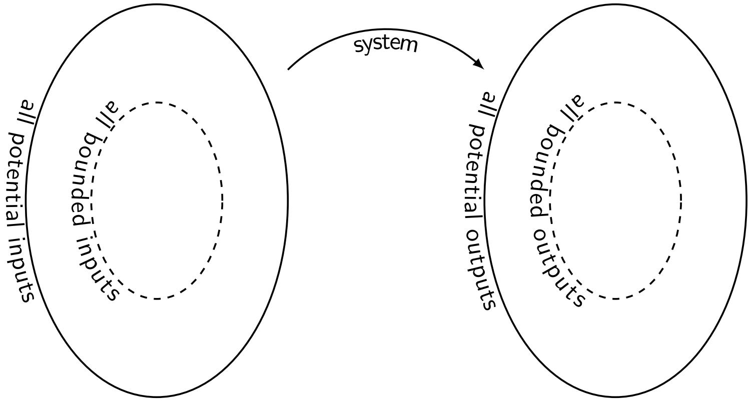


notes

- but some inputs get mapped into bounded forced responses

Visualizing this with avocados

for $\dot{v} = av + bu, a < 0$ all the bounded $u(t)$'s lead to bounded $v(t)$'s



notes

- and in this case all the bounded inputs get mapped into bounded forced responses

Important difference: the two systems have two different "stability" properties relative to how inputs get mapped into outputs!

notes

- indeed the two systems are intrinsically different

Remark

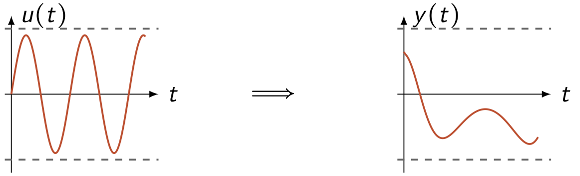
- "Stability" referring to specific equilibria:
  - simply stable equilibrium
  - convergent equilibrium
  - asymptotically stable equilibrium
- "Stability" referring to specific systems:
  - Bounded Input Bounded Output (BIBO) stable systems (we are seeing this now)
  - Input to State Stable (ISS) systems (we will not see this in this module)

notes

- more precisely be careful that there are several concepts of stability, these five being the most common ones in automatic control
- we will not see input to state stability, that you will instead see when you will do nonlinear systems in more details than what we do here

Definition: BIBO stability

If $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})$ is so that if $\boldsymbol{u}$ is bounded then $\boldsymbol{y}$ will be for sure bounded, then that system is said BIBO stable

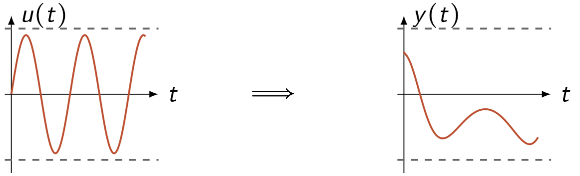


notes

- so let's consider a generic system, also a MIMO one, it does not matter
- what we saw in the previous examples is that sometimes a bounded input causes an unbounded output, while in others a bounded input causes only bounded outputs
- the mathematical definition of this property is then quite simple: consider an input, also time-varying. Are you able to find a bound to the input, i.e., say that it is max TOT and min TOT? If so, then the output shall also be limited, i.e., be max TOT2 and min TOT2 (note that these bounds may be different, and depending on the specific input, of course). If for any input that is bounded we know that the output will be bounded then the system is BIBO
- if I though find just one bounded input that produces an unbounded output, then the system is NOT BIBO

Definition: BIBO stability

If $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})$ is so that there exists at least one $\boldsymbol{u}$ bounded for which the corresponding $\boldsymbol{y}$ will be unbounded, then that system is said **not** BIBO stable



notes

- so let's consider a generic system, also a MIMO one, it does not matter
- what we saw in the previous examples is that sometimes a bounded input causes an unbounded output, while in others a bounded input causes only bounded outputs
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- if I though find just one bounded input that produces an unbounded output, then the system is NOT BIBO

Example

Is $\dot{y} = y + u$ BIBO stable?

- explain what BIBO stability means 14

notes

- no, and we can say this immediately because this is an unstable system

Example

Is $\dot{y} = yu$ BIBO stable?

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- no, and we can say this immediately because this is an unstable system

Example

Is $\dot{y} = u$ BIBO stable?

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notes

- no, and we can say this immediately because this is an unstable system

Example

Is $\dot{y} = -y$ BIBO stable?

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- no, this is not even non-autonomous! we cannot define the input output behavior here

Summarizing

Graphically explain the definition of BIBO (Bounded-Input Bounded-Output) stability and its connection to system behavior

Give examples of systems that are BIBO stable or not, and motivate their properties with physical intuitions

- if starting from the kernel of the input-avocado we are guaranteed to end up in the kernel of the output-avocado, then we have BIBO stability
- if not, not
- BIBO stability is a concept that relates to non-autonomous system - and it is not necessary that it it LTI, may also be nonlinear. But it needs to have at least one input

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

No dedicated python libraries for this ...

... but one can use the `control` library for checking the properties of the transfer function or impulse response of the system, if LTI of course

notes

- also this, to the best of our knowledge

Self-assessment material

notes

Question 1

Which of the following statements best describes BIBO (Bounded-Input Bounded-Output) stability?

Potential answers:

I: **(wrong)** A system is BIBO stable if all outputs remain constant regardless of the input.
II: **(correct)** A system is BIBO stable if every bounded input leads to a bounded output.
III: **(wrong)** A system is BIBO stable if it has at least one bounded output for an unbounded input.
IV: **(wrong)** A system is BIBO stable if it is asymptotically stable.
V: **(wrong)** I do not know

Solution 1: - explain what BIBO stability means 2

BIBO stability means that for any bounded input, the system output remains bounded. This definition does not require the system to be asymptotically stable.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

Which of the following systems is NOT BIBO stable?

Potential answers:

I: **(wrong)** $\dot{y} = -2y + u$.
II: **(correct)** $\dot{y} = y + u$.
III: **(wrong)** $\dot{y} = -y + 3u$.
IV: **(wrong)** $\dot{y} = -0.5y + u$.
V: **(wrong)** I do not know

Solution 1:

The system $\dot{y} = y + u$ has an unstable component, meaning that even bounded inputs can lead to unbounded outputs, making it NOT BIBO stable.

- explain what BIBO stability means 3

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

Which graphical interpretation correctly illustrates a system that is NOT BIBO stable?

Potential answers:

I: **(wrong)** A system where all bounded inputs correspond to bounded outputs.
II: **(correct)** A system where at least one bounded input results in an unbounded output.
III: **(wrong)** A system where all unbounded inputs lead to unbounded outputs.
IV: **(wrong)** A system where the impulse response is always decreasing over time.
V: **(wrong)** I do not know

Solution 1:

A system that is NOT BIBO stable has at least one bounded input that results in an unbounded output, violating the BIBO stability condition.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

A spacecraft is modeled as a single integrator $\dot{v} = bu$. What can be said about its BIBO stability?

Potential answers:

I: **(correct)** The system is NOT BIBO stable because a constant nonzero input leads to an unbounded velocity.
II: **(wrong)** The system is BIBO stable because velocity is a smooth function of time.
III: **(wrong)** The system is BIBO stable because acceleration remains bounded.
IV: **(wrong)** The system is BIBO stable because it eventually reaches a steady-state velocity.
V: **(wrong)** I do not know

Solution 1:

Since the velocity is the integral of the input, a constant nonzero input results in

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

Can one define the BIBO properties of an autonomous system?

Potential answers:

I: **(correct)** No, BIBO stability is defined in terms of input-output behavior.
II: **(wrong)** Yes, every system has a well-defined BIBO stability property.
III: **(wrong)** Only if the system is linear and time-invariant.
IV: **(wrong)** Yes, but only for discrete-time systems.
V: **(wrong)** I do not know

Solution 1:

BIBO stability concerns the boundedness of outputs given bounded inputs. Autonomous systems do not have external inputs, so BIBO stability is not directly applicable.

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notes

- see the associated solution(s), if compiled with that ones :)

Question 6

If a system has two inputs, can it be BIBO stable with respect to one input but not the other?

Potential answers:

I: **(correct)** Yes, different inputs can excite different dynamics in the system.
II: **(wrong)** No, BIBO stability is a system-wide property.
III: **(wrong)** Only if the system is nonlinear.
IV: **(wrong)** Only for discrete-time systems.
V: **(wrong)** I do not know

Solution 1:

BIBO stability is defined per input-output pair. A system can be stable for one input if that input does not excite unstable modes, while another input may lead to unbounded output.

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notes

- see the associated solution(s), if compiled with that ones :)

Question 7

BIBO stability properties are connected to equilibrium points, so if a system has three equilibria, do we need to analyze BIBO stability separately for each equilibrium?

Potential answers:

I: **(correct)** No, BIBO stability is an input-output property and does not depend on equilibria.
II: **(wrong)** Yes, because each equilibrium defines a different stability region.
III: **(wrong)** Only if the system is nonlinear.
IV: **(wrong)** Only for continuous-time systems.
V: **(wrong)** I do not know

Solution 1:

BIBO stability is defined for the entire systems input-output response, not per equilibrium. Stability around an equilibrium is a different concept (e.g., Lyapunov stability).

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notes

- see the associated solution(s), if compiled with that ones :)

Question 8

Is an integrator a BIBO stable system?

Potential answers:

I: **(correct)** No, an integrator accumulates input over time, leading to unbounded output for bounded input.
II: **(wrong)** Yes, because the output remains predictable.
III: **(wrong)** Only in discrete-time systems.
IV: **(wrong)** Only if the initial condition is zero.
V: **(wrong)** I do not know

Solution 1:

An integrator produces an output that grows indefinitely in response to a constant or oscillatory input, meaning it is not BIBO stable.

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notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module "explain what BIBO stability means"

- BIBO stability means "a bounded input must imply a bounded output"
- it is a concept that in general it is disconnected to that of marginal stability / convergence of an equilibrium

notes

- the most important remarks from this sub-module are these ones