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notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

explain and determine the convergence properties of an equilibrium

notes

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asymptotically stable equilibrium	u1, e1

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marginally stable equilibrium	u1, e1
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notes

Main ILO of sub-module "explain and determine the convergence properties of an equilibrium"

Graphically explain the definition of convergent equilibrium and interpret its practical meaning

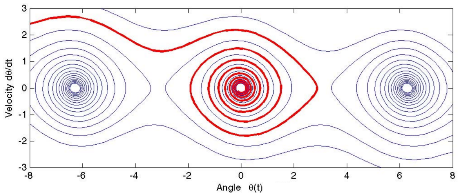
Give examples of systems that have convergent equilibria or not, and discuss scenarios where convergence is important

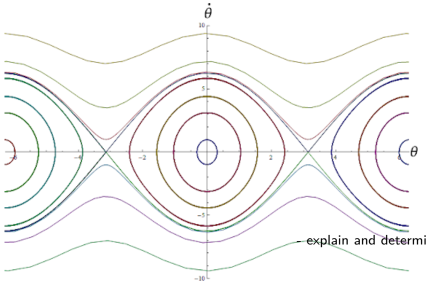
Determine if an equilibrium is convergent or not from inspecting a phase portrait, based on the flow near equilibrium

notes

- by the end of this module you shall be able to do this

Intuition: if I perturb a little bit this system from its equilibrium, where will it eventually end up?





notes

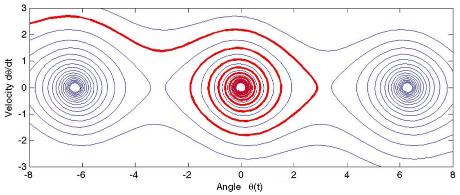
- this equilibrium is so that if I move the cart a bit from it, the cart won't go further away, and either return back there if there is some friction, or continuously oscillate if there is no friction

main point of convergence of an equilibrium: where does the system end up, eventually, if starting closeby?

notes

- this is the main concept that we are exploring
- studying the convergence of an equilibrium is crucial in practice because it ensures system predictability, stability, and safety. If an equilibrium is unstable, small disturbances can lead to catastrophic failures (e.g., in control systems or ecosystems). Understanding convergence helps in designing robust systems that maintain desired performance under real-world conditions

Convergent equilibrium (continuous time case, formally)



$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$ $\qquad \dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}_e\right)$ $\qquad \left(\boldsymbol{y}_e, \boldsymbol{u}_e\right) = \text{equilibrium}$

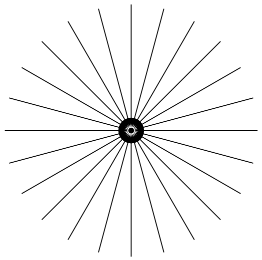
Definition (convergent equilibrium)

$\boldsymbol{y}_e$ = *is convergent if there is a neighborhood containing that equilibrium so that if one starts from within that neigborhood, eventually* $\boldsymbol{y}(t) \xrightarrow{t \to +\infty} \boldsymbol{y}_e$

notes

- now let's consider a property of trajectories 'at the end of time'
- more precisely, if a trajectory 'ends up' on an equilibrium it means that that equilibrium is attracting all the trajectories starting from initial conditions equals to the points in these trajectories
- imagine now that there exists an equilibrium for which if I start closeby, then at the end of time the trajectory starting from that closeby goes back to that equilibrium
- this formal definition then corresponds to say "if there exists at least one non-null-sized ball centered in an equilibrium for which all the trajectories that start in that ball end up at the end of time back to the equilibrium, then that equilibrium is said to be convergent"

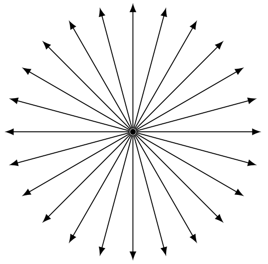
Some phase portraits to exemplify the potential situations



notes

- this situation is for which that equilibrium is convergent, since starting closeby will make the system eventually go there

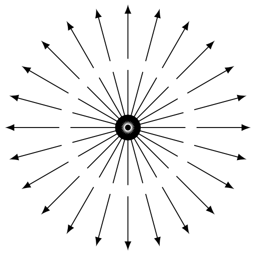
Some phase portraits to exemplify the potential situations



notes

- this situation is for which there seems to be no convergence properties for this equilibrium, since we cannot chose a neighborhood so that starting there we end up, at the end of time, back to the equilibrium

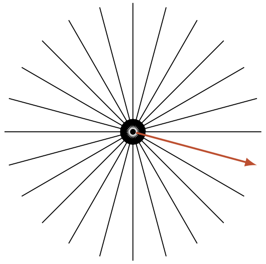
Some phase portraits to exemplify the potential situations



notes

- this situation is for which there is convergence, the important is to chose a small enough neighborhood so that the trajectories starting from inside there will eventually land on that equilibrium

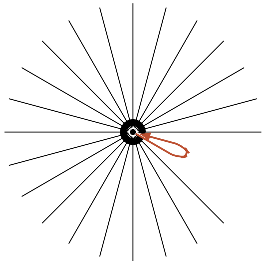
Some phase portraits to exemplify the potential situations



notes

- this situation is for which there is no convergence stability, since we have at least one trajectory that will escape, i.e., some situation for which we do not go back to the equilibrium

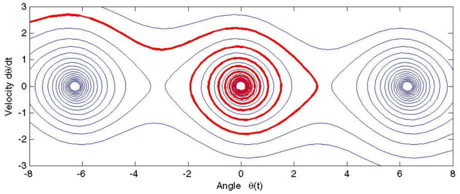
Some phase portraits to exemplify the potential situations



notes

- this situation is for which there is convergence, since even if we have some trajectories that will “escape” that equilibrium, actually that trajectories will bring eventually us back to the equilibrium and convergence cares only about what happens ‘at the end of time’

Important differences



- **simply stable equilibrium:** we can confine arbitrarily the trajectories around the equilibrium by reducing ε opportunely
- **convergent equilibrium:** we do not care about whether we can confine arbitrarily the trajectories or not; it may or not happen, it does not matter. We though know that if we start close enough then *eventually* the distance $\|\boldsymbol{y}(t) - \boldsymbol{y}_e\|$ will go to zero

notes

- note that convergency is a property of what happens "at the end of time", while simple stability means keeping track of what happens during the whole trajectory
- so they refer to very different properties

Discussion: Consider the continuous time system

$$\dot{y}(t) = \begin{cases} +y(t) & \text{if } t < 10^5 \\ -y(t) & \text{otherwise.} \end{cases}$$

Which type of equilibrium is 0? Possibilities:

- just simply stable
- just convergent
- both
- none
- I do not know

notes

- independently of where we start we eventually end up in 0, so it is a convergent equilibrium with a basin of attraction the whole $\mathbb{R}$. It is also simply stable equilibrium because for example if your opponent chooses $\varepsilon = 0.5$ then you can choose a very tiny δ-ball for which starting inside the δ-ball guarantees the trajectories staying in the ε-ball. So 0 is a marginally stable convergent equilibrium

Discussion: Consider the continuous time system

$$\dot{y}(t) = \begin{cases} +y(t) & \text{if } \sup\limits_{T \in (-\infty, t)} |y(T)| < 1 \\ -y(t) & \text{otherwise.} \end{cases}$$

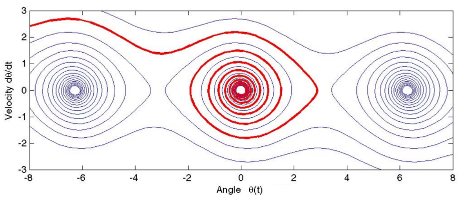
Which type of equilibrium is 0? Possibilities:

- just simply stable
- just convergent
- both
- none
- I do not know

notes

- independently of where we start we eventually end up in 0, so it is a convergent equilibrium with a basin of attraction the whole $\mathbb{R}$. It is not a simply stable (or marginally stable) equilibrium because for example if your opponent chooses $\varepsilon = 0.5$ then you cannot find any δ-ball for which starting inside the δ-ball guarantees the trajectories staying in the ε-ball. So 0 is actually an unstable but convergent equilibrium

Asymptotic stability



$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \overline{\boldsymbol{u}}\right) \qquad \boldsymbol{y}_e = \text{equilibrium}$$

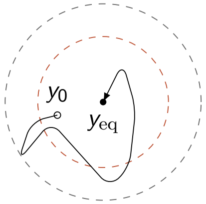
Definition (asymptotically stable equilibrium)

the equilibrium $\boldsymbol{y}_e$ is said to be asymptotically stable if it is simultaneously simply stable & convergent

notes

- asymptotic stability then means having both properties at the same time
- if we combine the two δ's, the one for simple stability and the one for convergence, that we know that they must exist by definition, and we take the smallest of these two δ's, then all the trajectories starting in this so-defined δ-ball are so that they both stay limited within the ε-ball and they eventually converge back to the equilibrium

Asymptotic stability in practice



Definition (asymptotically stable equilibrium)

the equilibrium $\boldsymbol{y}_e$ is said to be asymptotically stable if it is simultaneously simply stable & convergent

Discussion: how is the origin for a spring-mass system with friction? And the downwards-equilibrium for a pendulum with friction?

notes

- if you now think at a spring-mass system or a pendulum with friction it is obvious that these systems have equilibria that satisfy both these conditions: the trajectories do not "escape" (simple stability), plus they all end up in the original equilibrium because energy dissipates (convergence)

Summarizing

Graphically explain the definition of convergent equilibrium and interpret its practical meaning

Give examples of systems that have convergent equilibria or not, and discuss scenarios where convergence is important

Determine if an equilibrium is convergent or not from inspecting a phase portrait, based on the flow near equilibrium

- convergence deals with what happens at the end of time, while marginal stability deals with what happens in the transient
- an equilibrium is eventually convergent if it has a so-called 'basin of attraction', i.e., a zone of initial conditions in the phase portrait for which starting in that zone makes the free evolution end up in that equilibrium, eventually

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

No dedicated python toolbox for tihs

...but `scipy.linalg` can be used to analyse stability for LTI systems

notes

- (to the best of our knowledge)

Self-assessment material

notes

Question 1

If an equilibrium is not convergent, then is it necessarily unstable?

Potential answers:

I: **(correct)** Yes, non-convergent equilibria imply instability.
II: **(wrong)** No, an equilibrium can be non-convergent but still stable.
III: **(wrong)** No, all equilibria are inherently convergent.
IV: **(wrong)** It depends on whether the system is linear.
V: **(wrong)** I do not know

Solution 1:

If an equilibrium is not convergent, at least some trajectories do not remain nearby, indicating instability.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

If an equilibrium is convergent, does that mean it is marginally stable?

Potential answers:

I: **(wrong)** Yes, convergence always implies marginal stability.
II: **(correct)** No, convergence does not necessarily mean marginal stability.
III: **(wrong)** Only if the system has no damping.
IV: **(wrong)** Only in linear time-invariant systems.
V: **(wrong)** I do not know

Solution 1:

A convergent equilibrium may be asymptotically stable rather than marginally stable. Marginal stability requires bounded but non-decaying oscillations.

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

If an equilibrium is both convergent and marginally stable, then is it asymptotically stable?

Potential answers:

I: **(wrong)** Yes, marginal stability plus convergence implies asymptotic stability.
II: **(correct)** No, asymptotic stability requires trajectories to decay to equilibrium.
III: **(wrong)** Yes, unless external forces are applied.
IV: **(wrong)** Only in conservative systems.
V: **(wrong)** I do not know

Solution 1:

Marginal stability means trajectories remain bounded, but they do not necessarily decay to the equilibrium, which is required for asymptotic stability.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

If an equilibrium is marginally stable, then is it necessarily convergent?

Potential answers:

I: **(wrong)** Yes, marginal stability implies convergence.
II: **(correct)** No, marginal stability does not guarantee convergence.
III: **(wrong)** Only in discrete-time systems.
IV: **(wrong)** Only for systems with no external perturbations.
V: **(wrong)** I do not know

Solution 1:

Marginal stability only ensures trajectories remain bounded, but they may not necessarily approach the equilibrium.

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

Is the origin for the Lotka-Volterra model convergent?

Potential answers:

I: **(correct)** No, it is a saddle point and therefore unstable.
II: **(wrong)** Yes, because populations always return to equilibrium.
III: **(wrong)** Yes, because it has only non-positive eigenvalues.
IV: **(wrong)** It depends on the initial conditions.
V: **(wrong)** I do not know

Solution 1:

The origin in the Lotka-Volterra model is typically a saddle point, meaning small perturbations in certain directions grow, making it unstable.

notes

- see the associated solution(s), if compiled with that ones :)

Question 6

Is the non-null equilibrium for the Lotka-Volterra model asymptotically stable?

Potential answers:

I: **(wrong)** Yes, the non-null equilibrium always attracts trajectories.
II: **(correct)** No, it is typically neutrally stable with closed orbits.
III: **(wrong)** No, it is always unstable.
IV: **(wrong)** It depends on the values of the system parameters.
V: **(wrong)** I do not know

Solution 1:

The non-null equilibrium of the Lotka-Volterra model is usually neutrally stable, meaning trajectories form closed orbits around it rather than converging.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module "explain and determine the convergence properties of an equilibrium"

- convergence is disconnected from "marginal stability", since in general one may have one case and not the other, and viceversa, or both, or none
- the concept of convergence focuses on the limit behavior, ignoring the transient

notes

- the most important remarks from this sub-module are these ones