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notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

explain and determine the marginal stability of an equilibrium

notes

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Contents map

developed content units	taxonomy levels
marginally stable equilibrium	u1, e1
simply stable equilibrium	u1, e1

prerequisite content units	taxonomy levels
ODE	u1, e1
equilibrium	u1, e1

notes

Main ILO of sub-module "explain and determine the marginal stability of an equilibrium"

Graphically explain the definition of marginal stability for equilibria and provide graphical insights into its meaning

Identify and **give examples** of systems that have equilibria that are marginally stable or not, and relate these to real-world situations

Determine if an equilibrium is marginally stable or not by inspecting a phase portrait and by analyzing the behavior of the system near the equilibria

notes

- by the end of this module you shall be able to do this

Disclaimer

simply stable = marginally stable

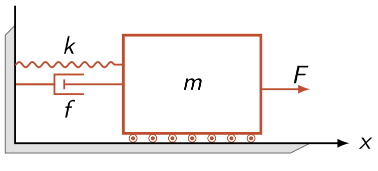
(they are synonyms)

notes

- don't get fooled by the names

Intuition: if I perturb a little bit this system from its equilibrium, will the system stay closeby or will it go resting to another place?

Example 1

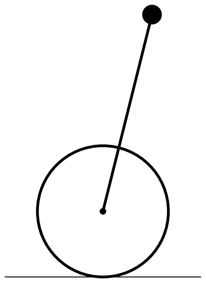


notes

- this equilibrium is so that if I move the cart a bit from it, the cart won't go further away, and either return back there if there is some friction, or continuously oscillate if there is no friction

Intuition: if I perturb a little bit this system from its equilibrium, will the system stay closeby or will it go resting to another place?

Example 2



notes

- if I don't use the motors, the upright equilibrium for this system is instead unstable, in the sense that if I move a bit the system from the equilibrium it will fall, i.e., go further away from it

Informal introduction to stability

- **asymptotically stable equilibrium:** if I perturb the equilibrium, the system will return there
- **marginally stable equilibrium:** if I perturb the equilibrium, the system will stay around there
- **unstable equilibrium:** if I perturb the equilibrium, the system will move away from it

mathematical definitions later on in the program!

notes

- so we can start also giving names to things: we will use these names with these intuitions
- actually later on we will introduce the concepts behind them more rigorously

stability = extremely important topic in automatic control!

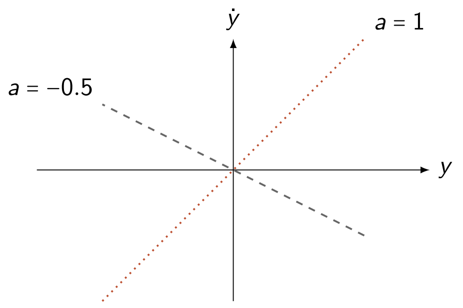
(e.g., will this nuclear plant blow up
if some disturbance slightly perturbs the operating point?)

will be analysed in more details later on in this module
and much more extensively in other courses
(feat. Lyapunov, Krasovskii, La-Salle among others)

notes

- once again, this module is on ODEs and not on stability; however better clarifying that this is an extremely important thing

Understanding the stability properties of $(y_0 = 0, u = 0)$ analysing $\dot{y} = ay = f(y)$ graphically

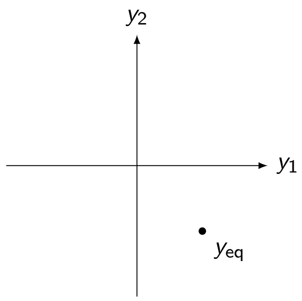


notes

- let's go though back to the autonomous version of the simplest system, also because we are analysing the case $u = 0$
- let's see what happens pretending we have two different f's, one given by the dotted line and one by the dashed one
- they map a positive y into a either positive or negative $\dot{y}$
- but if $y > 0 \implies \dot{y} > 0$ then y will become bigger and bigger, implying instability
- if instead $y > 0 \implies \dot{y} < 0$ then y will become smaller and smaller, implying convergence to zero (then in zero y stops moving)
- the same would happen if we were considering $y_0 < 0$
- this means that a plays a crucial role

This module = when is an equilibrium marginally stable

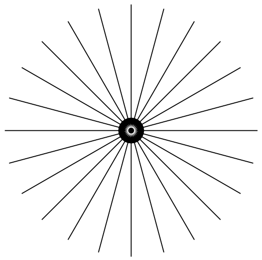
overarching question: are we able to bound the trajectories around that equilibrium?



notes

- consider a phase portrait, for which we consider essentially the trajectories of the autonomous system $\dot{y} = f(y, 0)$, or equivalently of the system with constant input $\dot{y} = f(y, u = \text{const.})$

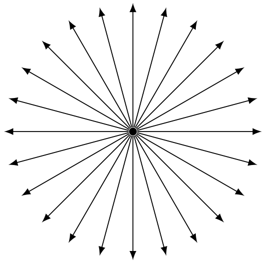
Some phase portraits to exemplify the potential situations



notes

- this situation is for which there is marginal stability, since if the 'boss' choses a neighborhood of the equilibrium, we can chose that neighborhood and keep the trajectories inside

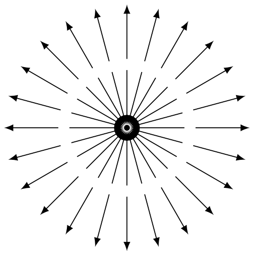
Some phase portraits to exemplify the potential situations



notes

- this situation is for which there is no marginal stability, since we cannot chose a neighborhood and keep the trajectories inside

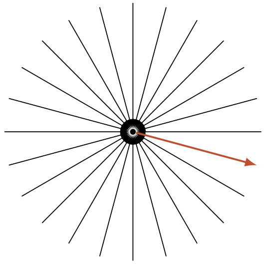
Some phase portraits to exemplify the potential situations



notes

- this situation is for which there is marginal stability, since if the ‘boss’ choses a neighborhood of the equilibrium that is big, we can in any case chose a small neighborhood and keep the trajectories inside

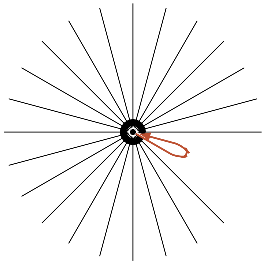
Some phase portraits to exemplify the potential situations



notes

- this situation is for which there is no marginal stability, since if the 'boss' choses a neighborhood of the equilibrium, we will have some trajectories that will escape that neighborhood

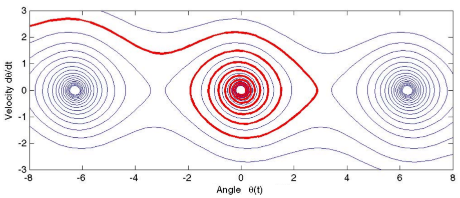
Some phase portraits to exemplify the potential situations



notes

- this situation is for which there is no marginal stability, since if the 'boss' choses a neighborhood of the equilibrium, we will have some trajectories that will escape that neighborhood. The fact is that the game has to hold for all the neighborhoods

Simply stable equilibrium (continuous time case, formally)



$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$ $\qquad \dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}_e)$ $\qquad (\boldsymbol{y}_e, \boldsymbol{u}_e) =$ equilibrium

Definition (simply stable equilibrium)

an equilibrium $\boldsymbol{y}_e$ is simply stable if for any chosen neighborhood of the equilibrium, there exists another one for which trajectories starting in the second neighborhood remain constrained in the first one for all time

notes

- so, formally, let's see the first type of equilibrium
- this reads as "$\boldsymbol{y}_e$ is simply stable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if the Euclidean distance between $\boldsymbol{y}_0$ and $\boldsymbol{y}_e$ is smaller or equal than δ then the Euclidean distance between $\boldsymbol{y}(t)$ and $\boldsymbol{y}_e$ will always be smaller than ε for all $t \geq 0$

A game to determine if $\boldsymbol{y}_e$ is simply stable or not

Players

you and an "opponent"

Game mechanics

- **step 1** your "opponent" draws a neighborhood containing $\boldsymbol{y}_e$, choosing any neighborhood he/she likes (small, big, whatever, but containing the equilibrium)
- **step 2** can you find a neighborhood within that neighborhood, again containing $\boldsymbol{y}_e$, so that if the trajectory starts from your neighborhood then it will stay also in the neighborhood of the opponent? *yes =* go back to step 1 and repeat, *no =* you lost and $\boldsymbol{y}_e$ is unstable
- **exiting the game** can you show that you will win step 2 for any neighborhood that the opponent can choose? Then the equilibrium is marginally stable

notes

- we can analyse what this means using a game whose purpose is this one
- the first step is this one. Note that you do not have control of what the opponent does. The opponent can choose any size of the ball centered in the equilibrium (the fact that it is centered there is important)
- you though know that as soon as your opponent chooses the ball, it chooses the set of trajectories that somehow "start" from that ball
- can you then find an inner ball for which you will stay within the outer ball? If no, then for sure you are dealing with an unstable equilibrium
- if you can play the game forever and get always "yes" then you won

important consequence: if an equilibrium is simply stable then it means that I can "confine" the trajectories in any ball of radius ε, with ε a users' choice

notes

- imagine this graphically: $\|\boldsymbol{y}(t) - \boldsymbol{y}_e\| \leq \varepsilon \quad \forall t \geq 0$ is the condition that says that the trajectory shall stay always within the ball chosen by the adversary $\exists \delta > 0$ s.t. if $\|\boldsymbol{y}_0 - \boldsymbol{y}_e\| \leq \delta$ then the trajectory stays within the outer ball is your move
- this is an important point: simply stable means that I can always find a delta-ball for which no trajectory that starts in the delta-ball goes outside of the epsilon-ball
- this means I can choose a set of initial conditions for which starting there I have confined trajectories

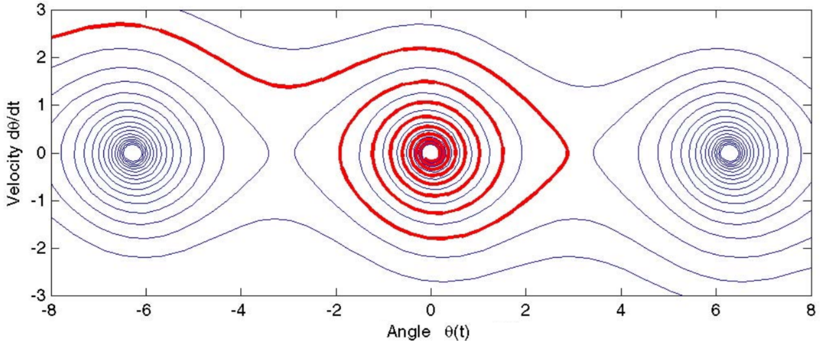
Definition of unstable equilibrium

equilibrium = "unstable" if not marginally stable

notes

- this follows from the definition too

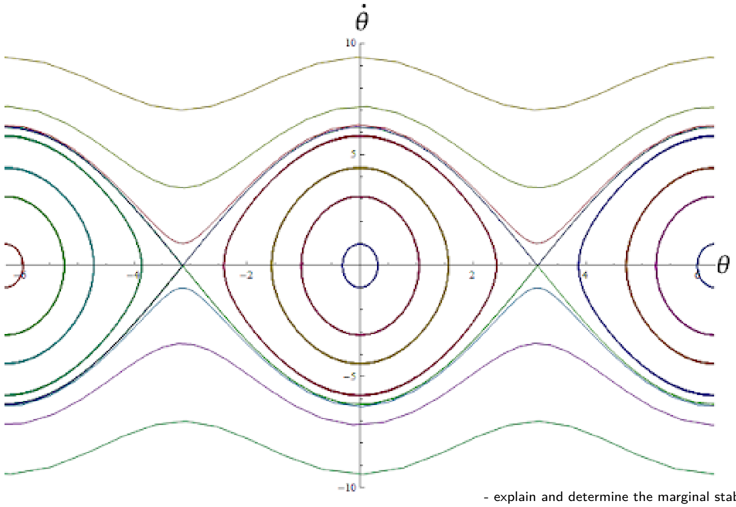
Example



notes

- here we see some stable and some unstable equilibria

Example



notes

- here we see some stable and some unstable equilibria

Summarizing

Graphically explain the definition of marginal stability for equilibria and provide graphical insights into its meaning

Identify and **give examples** of systems that have equilibria that are marginally stable or not, and relate these to real-world situations

Determine if an equilibrium is marginally stable or not by inspecting a phase portrait and by analyzing the behavior of the system near the equilibria

- one shall remember the 'game', and the order of who plays first, and who second
- once it is clear that as soon as there is at least one trajectory that "escapes" the equilibrium, that equilibrium is unstable, we got the main point

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

No dedicated python toolbox for tihs

...but `scipy.linalg` can be used to analyse stability for LTI systems

notes

- (to the best of our knowledge)

Self-assessment material

notes

Question 1

Does the concept of marginal stability of an equilibrium apply only to LTI systems?

Potential answers:

- I: **(wrong)** Yes, marginal stability is defined only for LTI systems.
- II: **(correct)** No, marginal stability can be defined for nonlinear systems as well.
- III: **(wrong)** Marginal stability is irrelevant for LTI systems.
- IV: **(wrong)** It only applies to mechanical systems.
- V: **(wrong)** I do not know

Solution 1:

Marginal stability is a property that can be analyzed for both LTI and nonlinear systems. While it is often introduced in the context of LTI systems, the concept extends to nonlinear systems under certain conditions.

- explain and determine the marginal stability of an equilibrium 2

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

Does the concept of marginal stability of an equilibrium apply only to continuous-time systems?

Potential answers:

- I: **(wrong)** Yes, marginal stability is only defined for continuous-time systems.
- II: **(wrong)** No, but it is more relevant in continuous-time systems.
- III: **(wrong)** No, discrete-time systems do not have equilibria.
- IV: **(correct)** No, marginal stability can be defined for both continuous and discrete-time systems.
- V: **(wrong)** I do not know

Solution 1:

- explain and determine the marginal stability of an equilibrium 3

Marginal stability applies to both continuous and discrete-time systems, though the definitions differ slightly in each case. In discrete-time systems, stability is typically assessed through eigenvalues inside the unit circle.

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

In the game of marginal stability, who starts? The boss or the apprentice?

Potential answers:

I: **(wrong)** The apprentice, since they test small perturbations.
II: **(correct)** The boss, since the system dynamics dictate the response.
III: **(wrong)** They both start at the same time.
IV: **(wrong)** There is no turn-based order in stability analysis.
V: **(wrong)** I do not know

Solution 1:

The system dynamics, dictated by the governing equations ("the boss"), determine how the state evolves. The "apprentice" (perturbations) follows.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

If a system has a marginally stable equilibrium, then all its equilibria must be marginally stable. Is this statement correct?

Potential answers:

I: **(correct)** No, stability properties are equilibrium-dependent.
II: **(wrong)** Yes, if one equilibrium is marginally stable, all others must be as well.
III: **(wrong)** The question is meaningless because marginal stability does not exist.
IV: **(wrong)** Only if the system is conservative.
V: **(wrong)** I do not know

Solution 1:

Each equilibrium must be analyzed individually, as stability depends on local properties of the system around each equilibrium.

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

Is the origin for the Lotka-Volterra model simply stable?

Potential answers:

I: **(correct)** No, it is a saddle point and therefore unstable.
II: **(wrong)** Yes, because populations always return to equilibrium.
III: **(wrong)** Yes, because it has only non-positive eigenvalues.
IV: **(wrong)** It depends on the initial conditions.
V: **(wrong)** I do not know

Solution 1:

The origin in the Lotka-Volterra model is typically a saddle point, meaning small perturbations in certain directions grow, making it unstable.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module "explain and determine the marginal stability of an equilibrium"

- marginal stability / simple stability is the property that answers the question "can I bound the evolutions, i.e., arbitrarily constrain them to do not get "too far" from an equilibrium by starting opportunely closeby the original equilibrium?
- an equilibrium is marginally stable or not depending on whether one is able to 'win' the 'choose your neighborhood' game
- phase portraits are very interpretable, to this regards
- there is a sort of "downgrading" phenomenon that happens here: one has to have *all* the trajectories behaving in a good way to have a certain property. One not behaving is enough for the "downgrading" of the equilibrium

notes

- the most important remarks from this sub-module are these ones