Systems Laboratory, Spring 2025

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notes

- welcome to the course!
- on this side of this document you will find notes that accompany the text typically visualized in class
- these notes are meant to convey the messages that are not displayed in the text on the side, and basically constitute what the teacher intends to say in class

Table of Contents I

notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

Connections between eigendecompositions and free evolution in continuous time LTI state space systems

notes

Contents map

developed content units	**taxonomy levels**
modal analysis	u1, e1

prerequisite content units	**taxonomy levels**
LTI ODE	u1, e1
state space system	u1, e1
eigenvalue	u1, e1
eigenspace	u1, e1
phase portrait	u1, e1

notes

Main ILO of sub-module
"Connections between eigendecompositions and free evolution in continuous time LTI state space systems"

Analyse the structure of the free evolution of the state variables by means of the eigendecomposition of the system matrix

notes

- by the end of this module you shall be able to do this

Important initial remark

focus = LTI in state space and free evolution, meaning $u(t) = 0$, and thus

$$\begin{cases} \dot{\boldsymbol{x}} & = A\boldsymbol{x} + Bu \\ y & = C\boldsymbol{x} \end{cases} \quad \mapsto \quad \begin{cases} \dot{\boldsymbol{x}} & = A\boldsymbol{x} \\ y & = C\boldsymbol{x} \end{cases}$$

notes

- the first simplification that we consider in this module is that we ignore the forced response

. . . and then an important disclaimer

$$\begin{cases} \dot{\boldsymbol{x}} &= A\boldsymbol{x} \\ y &= C\boldsymbol{x} \end{cases}$$

the module ignores what happens if A *is non-diagonalizable*

notes

- for the sake of aiding graphical intuition the module ignores what the visualizations should be in that cases where A is not diagonalizable (and thus should involve generalized eigenspaces - a concept that the reader may ignore what they are, for the sake of this course)

Roadmap

- set the focus just on $\boldsymbol{x}$, and not on $\boldsymbol{y}$
- get a graphical intuition of what $A\boldsymbol{x}$ means
- interpreting eigenspaces in the real of LTI continuous time systems
- adding the "superposition principle" ingredient to the mixture

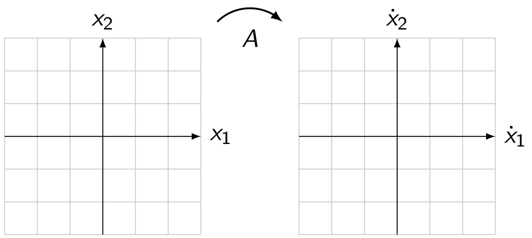
notes

- we will follow a specific pattern to get to the final point of the module

What does Ax mean, graphically?

notes

The physical meaning of the operation $\dot{\boldsymbol{x}} = A\boldsymbol{x}$



$\Longrightarrow$ structure of A determines how the time derivative $\dot{\boldsymbol{x}}$ is, and how the time derivative is determines the stability and time-evolution properties of the system. E.g.,

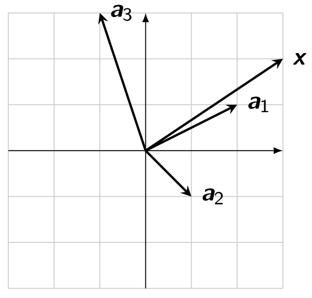
$$\operatorname{span}(A) = \begin{bmatrix} +1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad \text{if } x_1 \text{ grows then } x_2 \text{ diminishes, and viceversa}$$

notes

- then if we see that the columns of A generate this span, then we know that this must happen to the system
- recall that x_1 and x_2 are often physically interpretable variables, such as position and velocity. Being able to say sentences like this one means being able to describe in a qualitative way the mechanisms underlying the evolution of the system

How may we represent vectors and matrices?

$$\boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

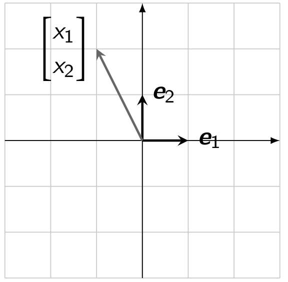


notes

- in the cartesian plane we can represent these objects as opportune (column) vectors
- note that due to our conventions we will draw a matrix A as a set of column vectors
- $\boldsymbol{a}_1$ is the first column, and so on

But what is a vector?

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2 = \boldsymbol{e}_1 x_1 + \boldsymbol{e}_2 x_2$$

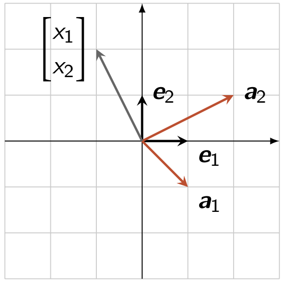


notes

- a vector is actually a combination of the elements of the canonical basis
- in a sense, the vector itself is defined by this basis
- also this concept will be expanded in later on courses . . .
- consider this also a sort of superposition of the effects

So, what is a matrix-vector product, geometrically?

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad A\boldsymbol{x} = ?$$



notes

- and this is a re-interpretation of the same thing we saw before
- matrix vector multiplication means multiplying each of the columns for the corresponding scalar, BUT this time we can see the whole operation
- the columns of the matrix are the transformed versions of the canonical basis, then the vector $[ab]^T$ goes expanding / compressing / flipping (depending on the values of its components) each of the transformed vectors independently

The effect of eigenspaces

notes

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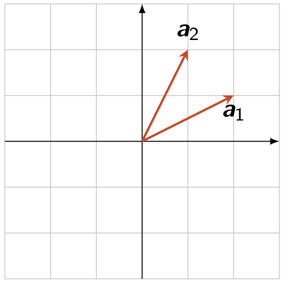
Eigenvectors of a square matrix



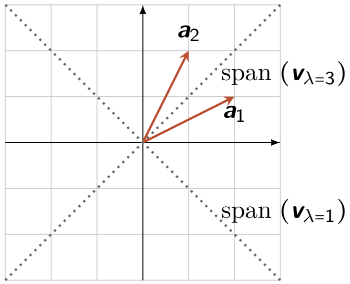
are there some directions that get only stretched, i.e., that do not rotate?

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mapsto \quad \boldsymbol{v}_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

notes

- note that we are defining things in a way for which this concept relates only to the case of square matrices
- the concept of eigenvector relates to the concept of warping the fabric of space
- if you warp space in this way as this matrix says, are there some 'lines' that remain untouched by the transformation? I.e., that do not rotate, but only get compressed or stretched?
- in this specific case there are two: the one defined by $[\alpha, \alpha]^T$, and the one defined by $[-\alpha, \alpha]^T$
- formally the question can be formulated in this way, where both λ and $\boldsymbol{x}$ are variables that shall be identified (i.e., read this as "for which $\boldsymbol{x}$ and α does this happen?")
- λ, the eigenvalues, should be interpreted as the "stretching factors", while $\boldsymbol{x}$ is any element within this "line that does not rotate"
- from the physical intuitions that we derive by looking at how the fabric of space warps, we get these two guesses
- putting these two guesses in the equation we see that they verify it, so they are actually the objects we were looking for

Eigenspaces = subspaces spanned by the eigenvectors-eigenvalues pairs



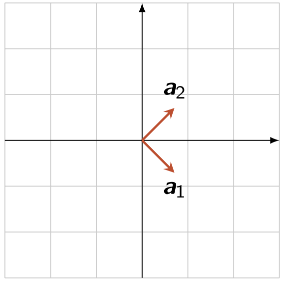
eigenspaces = subspaces spanned by the eigenvectors

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mapsto \quad \boldsymbol{v}_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

notes

- the eigenspaces are then that subspaces that are defined by the eigenvectors
- note that each eigenvalue has its own eigenvector – even if this is a bit imprecise; we will see the full picture when we discuss Jordan forms in a few units

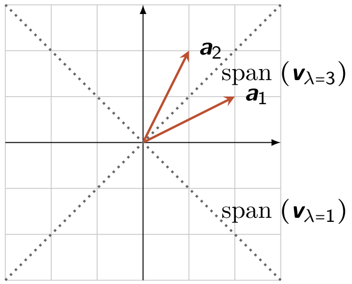
Eigenvectors: sometimes you may seem them from the transformation of the hypercube, sometimes you don't



notes

- there is a problem with visualizing eigenvalues, though: sometimes eigenvalues may be complex (something that is associated to rotation, as we will see soon)
- thus for matrices that perform rotations of the fabric the graphical approach seen before cannot apply
- very instrumental to understand why is the video from 3Blue1Brown `https://www.youtube.com/watch?v=v0YEaeIClKY&ab_channel=3Blue1Brown`

Why do we like eigenspaces?

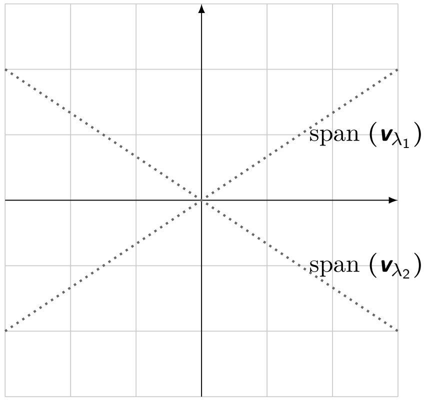


because $\dot{\boldsymbol{x}} = \lambda \boldsymbol{x} \implies$ "keep moving along that line"

notes

- the eigenspaces are so that if the initial condition of the system is on that subspace, then the direction of motion is aligned with that subspace, and this means that the system will stay there

Why do we like eigenspaces? Take 2

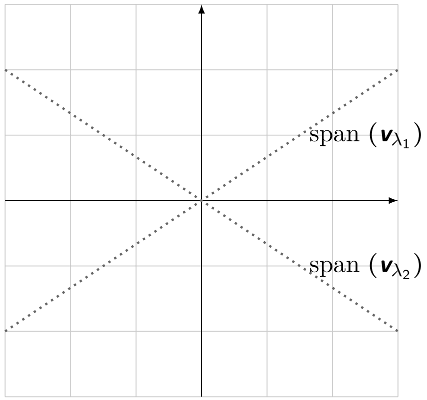


superposition principle $\Longrightarrow$ one can caracterize the whole phase portrait

notes

- moreover the eigenvectors may define also where the system will end up in free evolution starting from a generic point
- this concept is connected with the one of “mode of a system”

Why do we like eigenspaces? Take 3

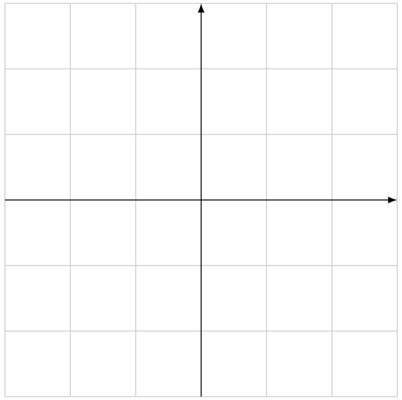


the trajectory along each eigenspace is driven by a first order differential equation
$\Longrightarrow$ if $\boldsymbol{x}_0 \in \mathrm{span}(\boldsymbol{v}_\lambda)$, then $\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{x}_0$

notes

- finally we get that the movement along each specific eigenspace is essentially like $\dot{y} = \alpha y$, and thus dominated by exponentials

Examples



notes

- likely best to watch the video of the lecture here, that there is going to be some examples of how different eigenvalues and eigenspaces will imply different phase portraits

How do we compute eigenvalues and eigenvectors numerically?

```
eigenvalues, eigenvectors = numpy.linalg.eig(A)
```


notes

- computationally speaking, all these things have been implemented in very robust and user friendly libraries

Summarizing

Analyse the structure of the free evolution of the state variables by means of the eigendecomposition of the system matrix

- find the eigenspaces and the eigenvalues
- depending on the values of the eigenvalues, understand how the trajectories along the eigenspaces look like
- depending on the relative angle among the eigenspaces, infer the phase portrait
- if the system matrix is not diagonalizable, then this concept complicates due to the presence of generalized eigenspaces (not in this module)

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

Linear algebra in general

https://numpy.org/doc/2.1/reference/routines.linalg.html

notes

- this library can be used to do much more than eigendecompositions

Self-assessment material

notes

Question 1

What does a positive eigenvalue imply about the system's behavior along its corresponding eigenspace?

Potential answers:

I: **(correct)** The state grows exponentially along that eigenspace.
II: **(wrong)** The state decays exponentially along that eigenspace.
III: **(wrong)** The state oscillates along that eigenspace.
IV: **(wrong)** The state remains constant along that eigenspace.
V: **(wrong)** I do not know.

Solution 1:

A positive eigenvalue implies that the state grows exponentially along the corresponding eigenspace. This is derived from the solution $\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{x}_0$, where $\lambda > 0$ leads to exponential growth.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

In the context of free evolution of a linear time-invariant (LTI) system, what does the equation $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ represent?

Potential answers:

I: **(wrong)** The evolution of the system's output over time.
II: **(correct)** The evolution of the state variables over time, influenced by the system matrix A.
III: **(wrong)** The relationship between input and output signals in the system.
IV: **(wrong)** The response of the system to external inputs.
V: **(wrong)** I do not know

Solution 1:

The equation $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ describes the free evolution of the system's state variables over time, where the rate of change of the state vector $\boldsymbol{x}$ is determined by the system matrix A.

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

Why is it useful to consider the eigendecomposition of the system matrix A in analyzing the free evolution of state variables?

Potential answers:

I: **(wrong)** It simplifies calculating the system's forced response.
II: **(wrong)** It directly determines the output y of the system.
III: **(correct)** It helps identify invariant directions (eigenvectors) and growth/decay rates (eigenvalues) that govern the system's behavior over time.
IV: **(wrong)** It only affects the graphical representation, not the actual system behavior.
V: **(wrong)** I do not know

Solution 1:

Eigendecomposition reveals the system's eigenvectors and eigenvalues, which represent invariant directions and the associated rates of exponential growth or decay. This insight simplifies the analysis of the system's dynamics.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

In a graphical representation, what does the matrix-vector product $A\boldsymbol{x}$ illustrate in the context of system dynamics?

Potential answers:

I: **(wrong)** The projection of the state vector onto the output space.
II: **(wrong)** The response of the system to a unit impulse.
III: **(correct)** Where the trajectory of the system is going, starting from $\boldsymbol{x}$.
IV: **(wrong)** The change in the input signal over time.
V: **(wrong)** I do not know

Solution 1:

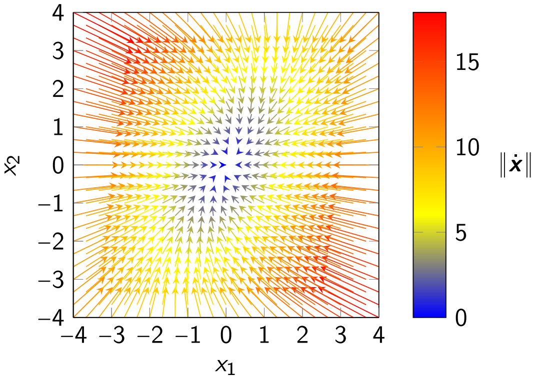
The product $A\boldsymbol{x}$ represents $\dot{\boldsymbol{x}}$, that indicates the system's dynamics on the state evolution.

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

Which eigenvalues and eigenspaces would you say characterize the system matrix A, looking just at this phase portrait?



notes

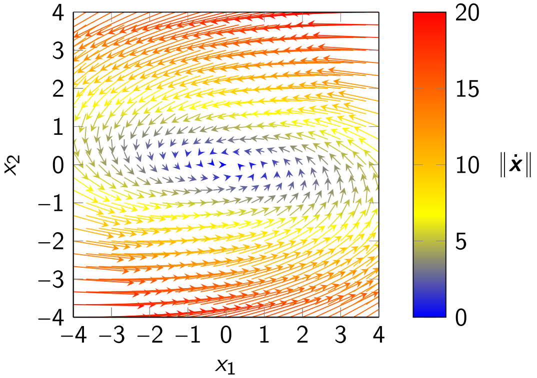
- see the associated solution(s), if compiled with that ones :)

Solution 1:

The eigenspaces are associated to that subspaces identified by a series of aligned quivers. The eigenvalues are positive or negative depending on the movement. If there are complex eigenvalues then there are spiral like movements.

Question 6

Which eigenvalues and eigenspaces would you say characterize the system matrix A, looking just at this phase portrait?



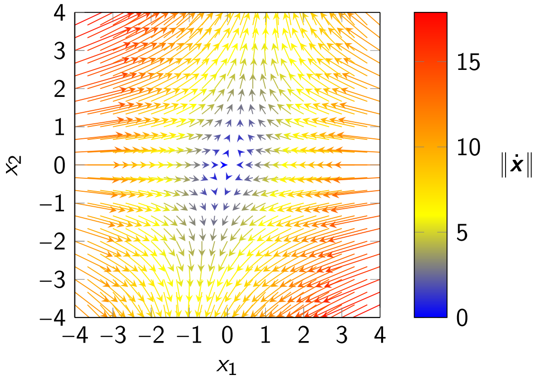
notes

- see the associated solution(s), if compiled with that ones :)

Solution 1:

Question 7

Which eigenvalues and eigenspaces would you say characterize the system matrix A, looking just at this phase portrait?



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Solution 1:

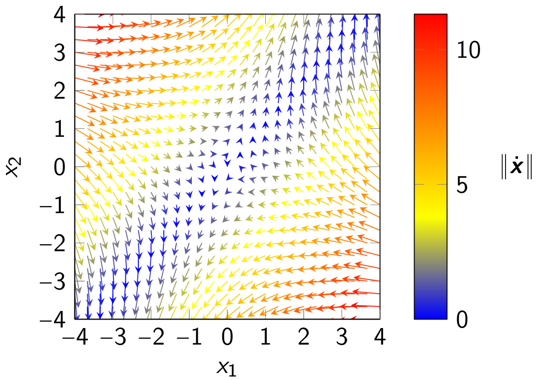
The eigenspaces are associated to that subspaces identified by a series of aligned quivers. The eigenvalues are positive or negative depending on the movement. If there are complex eigenvalues then there are spiral like movements.

notes

- see the associated solution(s), if compiled with that ones :)

Question 8

Which eigenvalues and eigenspaces would you say characterize the system matrix A, looking just at this phase portrait?



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notes

- see the associated solution(s), if compiled with that ones :)

Solution 1:

Recap of sub-module
"Connections between eigendecompositions and free evolution in continuous time LTI state space systems"

- the eigenvalues of the system matrix A give the growth / decay rates of the modes $e^{\alpha t}$ of the free evolution of the system
- along eigenspaces, the trajectory of the free evolution is "simple", i.e., aligned with that eigenspace
- the kernel of the system matrix gives us the equilibria corresponding to $u = 0$

notes

- the most important remarks from this sub-module are these ones