Systems Laboratory, Spring 2025

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notes

- welcome to the course!
- on this side of this document you will find notes that accompany the text typically visualized in class
- these notes are meant to convey the messages that are not displayed in the text on the side, and basically constitute what the teacher intends to say in class

Table of Contents I

notes

- this is the table of contents of this document; each section corresponds to a specific part of the course state space representations

notes

Contents map

developed content units	taxonomy levels
state of a system	u1, e1
separation principle	u1, e1

prerequisite content units	taxonomy levels
ODE	u1, e1

notes

Main ILO of sub-module "state space representations"

Define the meaning of "state space representation" in the context of linear and non-linear dynamical systems

notes

- by the end of this module you shall be able to do this

Discussion: which information do you need to forecast accurately how long you may use your cellphone before its battery hits 0%?

notes

- think at which factors are important for you

Summarizing

these pieces of information contain all I need to forecast the future evolution of the battery level:

- current level of charge of the battery
- how much I will use the phone in the future
- how healthy the battery of my phone is
- which environmental factors may induce additional effects (too warm, too cold)

notes

- the state condenses somehow the past
- from a control point of view this is important, because the state somehow works as a "memory": to decide wich u is best right now, I just need to check what is the current state – I do not care about what state the system was experiencing before
- so this is a concept that is very instrumental for control

A simple model of the battery charge as a dynamical system

$$\text{Time Remaining} = \frac{\text{Discharge Rate}}{\text{Remaining Capacity}} \qquad \text{example: } \frac{500\mathrm{mA}}{2000\mathrm{mAh}} = 4\text{hours}$$

rewriting as an ODE:

- $y(t) = Q(t)$ = remaining battery capacity at time t (mAh)
- $u(t) = I(t)$ = current discharge rate at time t (mA)

$$\implies \quad \dot{y} = -u$$

notes

- let's make a physical model of this

What is a state?

$$\begin{cases} \dot{x} & = -u \\ y = x \end{cases}$$

"the current value of the state $x(t)$ contains all the information necessary to forecast the future evolution of the output $y(t)$ and of the state $x(t)$, assuming to know the future $u(t)$. I.e., to compute the future values $y(t+\tau)$ and $x(t+\tau)$ it is enough to know the current $x(t)$ and the current and future inputs $u(t:t+\tau)$"

notes

- the state condenses somehow the past
- from a control point of view this is important, because the state somehow works as a "memory": to decide wich u is best right now, I just need to check what is the current state – I do not care about what state the system was experiencing before
- so this is a concept that is very instrumental for control

Question 1

In a spring-mass system, which of the following is a valid state variable?

Potential answers:

I: **(wrong)** The temperature of the spring.
II: **(correct)** The displacement of the mass from its equilibrium position.
III: **(wrong)** The color of the mass.
IV: **(wrong)** The external force applied to the system.
V: **(wrong)** I do not know.

Solution 1:

The displacement of the mass from its equilibrium position is a valid state variable because it describes the system's configuration and is essential for predicting its future behavior. Temperature and color are irrelevant, and the external force is an input, not a state.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

Which of the following pairs of variables can fully describe the state of a spring-mass system?

Potential answers:

I: **(wrong)** The mass of the spring and the stiffness of the mass.
II: **(wrong)** The external force and the displacement of the mass.
III: **(correct)** The displacement of the mass and the velocity of the mass.
IV: **(wrong)** The acceleration of the mass and the color of the spring.
V: **(wrong)** I do not know.

Solution 1:

The displacement of the mass and the velocity of the mass fully describe the state of a spring-mass system because they capture the system's current configuration (displacement) and its rate of change (velocity). Mass, stiffness, external force, and color are not state variables.

notes

- see the associated solution(s), if compiled with that ones :)

What do we mean with "modelling a state-space dynamical system"?

Defining

$$\begin{cases} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\theta}\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\theta}\right) \end{cases}$$

and

- the variables
  - $\boldsymbol{u}$ = the inputs
  - $\boldsymbol{d}$ = the disturbances
  - $\boldsymbol{x}$ = the states vector
  - $\boldsymbol{y}$ = the measured outputs
- the structure of the functions $\boldsymbol{f}$ and $\boldsymbol{g}$
- the value of the parameters $\boldsymbol{\theta}$

notes

- these are called state space representations
- take home message: the input-output maps saw in other modules are not the unique ways of representing systems

State space model - definition

Ingredients:

- the number of inputs, outputs and state variables must be finite
- the differential equations must be first order
- the separation principle *(the current value of the state contains all the information necessary to forecast the future evolution of the outputs and of the state)* shall be satisfied

state space model = finite set of first-order differential equations that connect a finite set of inputs, outputs and state variables so that they satisfy the separation principle

notes

- so, if we recall what we did in some modules ago, this was the formal definition of a state space system
- remember that, first of all, it is a finite representation: for example a metal bar that is heating up, we may describe it with partial differential equations. But this would mean considering the temperature in every point, and this means an infinite number of points - no good
- we work with computers, and somehow we need always to consider a discrete and finite number of objects. Thus we consider finite number of states

State space representations - Notation

- $u_1, \ldots, u_m$ = inputs
- $x_1, \ldots, x_n$ = states
- $y_1, \ldots, y_p$ = outputs
- $d_1, \ldots, d_q$ = disturbances

notes

- remember also that this is the standard notation (but the q in d_q, for which there is no standard notation)
- if you will use something different in your job you will look like a fool

State space representations - Notation

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})\end{aligned}$$

- $\boldsymbol{f}$ = state transition map
- $\boldsymbol{g}$ = output map

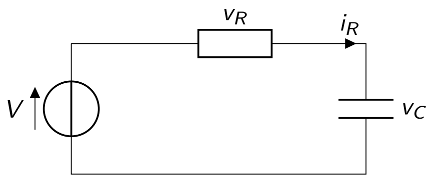
notes

- finally, we use this notation

examples

notes

RC-circuit



$$\dot{v}_C = -\frac{1}{RC} v_C + \frac{1}{RC} V \tag{1}$$

or, using control-oriented names,

$$\dot{x} = -\frac{1}{RC} x + \frac{1}{RC} u \qquad y = x \tag{2}$$

notes

- we start with the simplest dynamic system possible, that is a scalar first order system where the dynamics are implied by the presence of the capacitor
- note that if we were not having the capacitor the system would have been a static one
- here we can change the names so to highlight what is the output and the input, i.e., what we can steer
- here note that how much y grows depends on both y and u, and this dependence is "fixed" by R and C
- qualitatively what happens if we put $u = 0$ but we have some initial tension?

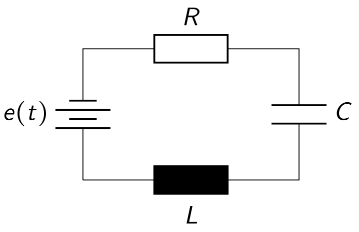
Generalization: exponential growth, scalar version

$$\dot{x} = \alpha x + \beta u \qquad y = x \tag{3}$$

notes

- the previous example can be generalized in this way
- we will see better later on that "exponentials" play a big role here, since if we neglect u you see that we have that we must have that the derivative of y must be proportional to y itself. Exponentials have this property (also sinusoids, but we know that sinusoids are complex exponentials, because of Euler's identities)
- we will see this better later on though
- here note that how much y grows depends on both y and u, and this dependence is "fixed" by α and β

Generalizing in an other way: RCL-circuits



EOM: Kirchhoff laws $\Longrightarrow$ $v_L(t) = L\frac{di(t)}{dt}$ $\quad v_i(t) = Ri(t)$ $\quad v_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$

$$e(t) = v_L(t) + v_R(t) + v_C(t) \quad \Longrightarrow \quad e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau \tag{4}$$

notes

- what happens if we add an inductor?
- the equations of motion can be derived from Kirchhoff's laws, that can be summarized in this way
- and then we can state that the tension in the generator must equal to the sum of the tensions along the various components

Generalizing in an other way: RCL-circuits part two

$$e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau \tag{5}$$

can be rewritten as

$$\begin{cases} \overset{\cdot}{\left(\int_0^t i(\tau)d\tau\right)} & = i(t) \\ \dot{i}(t) & = \frac{1}{L}e(t) - \frac{R}{L}i(t) - \frac{1}{LC}\int_0^t i(\tau)d\tau \end{cases} \tag{6}$$

that can be rewritten as

$$\begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = \frac{1}{L}u(t) - \frac{R}{L}x_2 - \frac{1}{LC}x_1 \end{cases} \qquad y = x_2 \tag{7}$$

notes

- for the purposes of the course it is convenient to do this rewriting
- and then this second rewriting, where we express the variables as states
- somehow it may have been more convenient to write x instead of y, but this is a sort of nuisance that will not matter at all when you understand the messages from this course

Exponential growth, matricial version

Generalization of all linear systems, thus also of "RCL circuits"

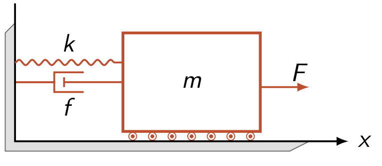
$$\begin{cases} \dot{\boldsymbol{x}} & = A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} & = C\boldsymbol{x} \end{cases}$$

notes

- this example generalizes the one saw before. Better to recall the geometrical interpretation of a matrix times a vector, that highlights each column of A to be a direction in the space where $\dot{\boldsymbol{x}}$ lives, and every component of $\boldsymbol{x}$ being thus how much that direction of that column should be followed
- the same interpretation of columns times scalars follows for the term $B\boldsymbol{u}$. Here each term of $\boldsymbol{u}$
- also for this type of ODE we will have that exponentials play a big role

Spring-mass systems

E.g., position of a cart fastened with a spring to a wall and subject to friction



EOM:

- force from the spring: $F_x(t) = -kx(t)$
- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

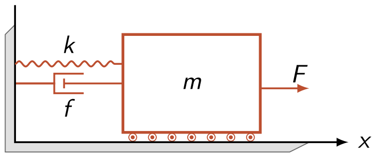
$$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t) \qquad \mapsto \qquad m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + F(t)$$

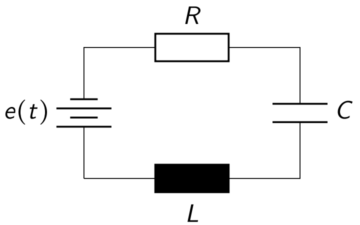
(rewritable again as $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$)

notes

- next example: a cart
- Newton's laws of motion tell us the following
- and we can rewrite things again in this way
- note that this is thus another example of the linear system that we used to represent RCL circuits

Important message: these two systems are "the same"





thus studying $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ in means studying both systems simultaneously!

notes

- so we can conclude that essentially these two systems share the same structure, only with different parameters
- so this means that studying the equations in general makes us "save time", because we study all these systems (and many more) in one shot

Lotka-Volterra

- $y_{\text{prey}} :=$ prey
- $y_{\text{pred}} :=$ predator

$$\begin{cases} \dot{y}_{\text{prey}} & = \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} & = -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

`./LotkaVolterraSimulator.ipynb`

notes

- of course not all the systems are of the type $\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$. This one is for example nonlinear, since there is a product among the y's that cannot be captured by the linear relation above
- this example was seen before. More information and history behind it in `https://en.wikipedia.org/wiki/Lotka%E2%80%93Volterra_equations`
- go through the python notebook for more information

Van-der-Pol oscillator

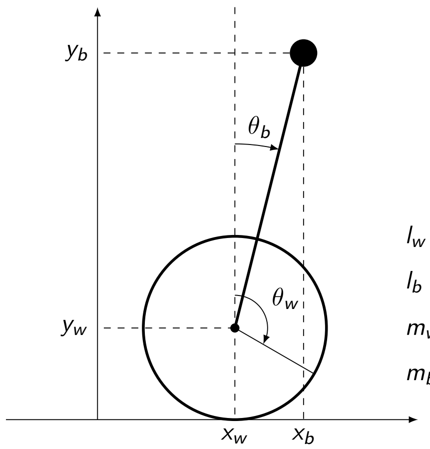
$$\begin{cases} \dot{x}_1 &= \mu \left(x_1 - \frac{x_1^3}{3} - x_2 \right) \\ \dot{x}_2 &= \frac{x_1}{\mu} \end{cases} \tag{8}$$

`./VanDerPolSimulator.ipynb`

notes

- another interesting example is the oscillator here, that we will see in more details later on
- the nice property of this system is that it has an orbit that attracts all the remaining ones
- we will discuss this example a few times. If you are already now interested in reading about it, check `https://en.wikipedia.org/wiki/Van_der_Pol_oscillator`
- go through the python notebook for more information

Balancing robot



l_w = radius of the wheel

l_b = body-wheel's center distance

m_w = mass of the wheel

m_b = mass of the body

notes

- another example is a model of a segway
- here we will use this notation

Balancing robot

$$
\begin{aligned}
\left(I_b + m_b l_b^2\right)\ddot{\theta}_b &= +m_b l_b g \sin\left(\theta_b\right) - m_b l_b \ddot{x}_w \cos\left(\theta_b\right) - \frac{K_t}{R_m} v_m + \left(\frac{K_e K_t}{R_m} + b_f\right)\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \\
\left(\frac{I_w}{l_w} + l_w m_b + l_w m_w\right)\ddot{x}_w &= -m_b l_b l_w \ddot{\theta}_b \cos\left(\theta_b\right) + m_b l_b l_w \dot{\theta}_b^2 \sin\left(\theta_b\right) + \frac{K_t}{R_m} v_m - \left(\frac{K_e K_t}{R_m} + b_f\right)\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right)
\end{aligned}
\tag{9}
$$

notes

- the EOM can be found again with Newton's laws
- this is a nonlinear system, since it contains some trigonometric transformations of the variables

Insulin concentration

- $x_1 :=$ sugar concentration
- $x_2 :=$ insulin concentration
- $u_1 :=$ food intake
- $u_2 :=$ insulin intake
- $c :=$ sugar concentration in fasting *(person-specific)*

$$
\begin{cases}
\dot{x}_2 = a_{21}\left(x_1 - c\right) - a_{22} x_2 + b_2 u_2 & x_1 \geq c \\
\dot{x}_2 = -a_{22} x_2 + b_2 u_2 & x_1 < c
\end{cases}
$$

$$
\begin{cases}
\dot{x}_1 = -a_{11} x_1 x_2 - a_{12}\left(x_1 - c\right) + b_1 u_1 & x_1 \geq c \\
\dot{x}_1 = -a_{11} x_1 x_2 + b_1 u_1 & x_1 < c
\end{cases}
$$

notes

- this is a switched system that represents in a very simplified way what happens to the body when eating or taking artificial insuline
- depending on whether there is more or less sugar in the blood than what is the person specific parameter c, then the body answers in different ways
- the main take home message for this model is that it tries to mimick biological phenomena that are quite understood
- the model can then be used to design when / how much to eat and to inject insuline
- for more information towards biology see for example `https://en.wikipedia.org/wiki/Insulin`, while for control-oriented explanations see for example "Model individualization for artificial pancreas", in Computer Methods and Programs in Biomedicine, Volume 171, April 2019, Pages 133-140, Messori et al.

Summarizing

Define the meaning of "state space representation" in the context of linear and non-linear dynamical systems

- recall the definition of state space model
- be sure to have interiorized the separation principle with some practical examples

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

Important library

https://python-control.readthedocs.io/en/0.10.1/conventions.html#state-space-systems

notes

- this is more or less a reference library

Self-assessment material

notes

Question 3

What is the primary purpose of the separation principle in state space representations?

Potential answers:

I: **(wrong)** To ensure that the system has an infinite number of states.
II: **(wrong)** To eliminate the need for inputs in the system model.
III: **(correct)** To ensure that the current state contains all information needed to predict future behavior.
IV: **(wrong)** To simplify the computation of system eigenvalues.
V: **(wrong)** I do not know.

Solution 1:

The separation principle ensures that the current state contains all the information necessary to predict the future evolution of the system, given the future inputs. This is a fundamental property of state space representations.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

Which of the following is a valid state variable in a state space representation of a dynamical system?

Potential answers:

I: **(wrong)** The external force applied to the system.
II: **(correct)** The displacement of a mass in a spring-mass system.
III: **(wrong)** The color of the system components.
IV: **(wrong)** The temperature of the environment.
V: **(wrong)** I do not know.

Solution 1:

The displacement of a mass in a spring-mass system is a valid state variable because it describes the system's configuration and is essential for predicting its future behavior. External forces, color, and environmental temperature are not state variables.

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

What does the state transition map $\boldsymbol{f}$ in a state space representation describe?

Potential answers:

I: **(wrong)** The relationship between inputs and outputs.
II: **(correct)** The evolution of the state variables over time.
III: **(wrong)** The effect of disturbances on the system.
IV: **(wrong)** The stability of the system.
V: **(wrong)** I do not know.

Solution 1:

The state transition map $\boldsymbol{f}$ describes how the state variables evolve over time based on the current state and inputs. It is a key component of state space representations.

notes

- see the associated solution(s), if compiled with that ones :)

Question 6

What is the role of the output map $\boldsymbol{g}$ in a state space representation?

Potential answers:

I: **(wrong)** To define the system's stability.
II: **(wrong)** To describe the evolution of the state variables.
III: **(correct)** To relate the state variables and inputs to the measured outputs.
IV: **(wrong)** To eliminate the need for disturbances in the model.
V: **(wrong)** I do not know.

Solution 1:

The output map $\boldsymbol{g}$ relates the state variables and inputs to the measured outputs. It defines how the system's internal state is reflected in the observable outputs.

notes

- see the associated solution(s), if compiled with that ones :)

Question 7

Which of the following pairs of variables is sufficient to describe the state of a simple pendulum system?

Potential answers:

I: **(wrong)** The mass of the pendulum and the length of the string.
II: **(wrong)** The external torque and the angular displacement.
III: **(correct)** The angular displacement and the angular velocity.
IV: **(wrong)** The color of the pendulum and the gravitational constant.
V: **(wrong)** I do not know.

Solution 1:

The angular displacement and the angular velocity are sufficient to describe the state of a simple pendulum system because they capture the system's current configuration (displacement) and its rate of change (velocity). Mass, length, external torque, and color are not state variables.

notes

- see the associated solution(s), if compiled with that ones :)

Exercise: find which parts of these paragraphs are correct and which ones are wrong

The RCL circuit can be modeled by a second-order linear differential equation where the inductance, resistance, and capacitance determine the system's resonance frequency. Interestingly, in an underdamped RCL circuit, the system will always return to equilibrium without oscillating, which reflects the energy dissipation in the resistor.

notes

- the solution is:
- RCL Circuit Misconception: The statement "the system will always return to equilibrium without oscillating" is incorrect. An underdamped RCL circuit does oscillate before eventually returning to equilibrium due to the resistance.

Exercise: find which parts of these paragraphs are correct and which ones are wrong

The Lotka-Volterra model is a non-linear system that describes interactions between two species: one as a predator and the other as prey. The model assumes that the growth rate of the prey population is proportional to the current population size, which would mean that the population would grow indefinitely in the absence of predators. Similarly, the predator population is dependent solely on the availability of prey, implying that predators could not survive without prey even if there were other food sources available.

notes

- the solution is:
- Lotka-Volterra Misconception: The claim that "predators could not survive without prey even if there were other food sources available" oversimplifies the model. The model assumes that the prey is the only food source, but in reality, predators might have alternative food sources.

Exercise: find which parts of these paragraphs are correct and which ones are wrong

The Van der Pol oscillator is an example of a non-linear system that exhibits limit cycle behavior. This behavior is critical as it shows how the system can maintain a stable oscillation regardless of initial conditions, which is a feature not present in linear oscillators. It's important to note that the Van der Pol oscillator can only have a single limit cycle, and any perturbations will lead to a quick return to this cycle, indicating that the system is highly stable.

notes

- the solution is:
- Van der Pol Oscillator Misconception: The statement "the Van der Pol oscillator can only have a single limit cycle" is correct, but saying that "any perturbations will lead to a quick return to this cycle, indicating that the system is highly stable" is misleading. The Van der Pol oscillator returns to its limit cycle, but the speed and nature of this return depend on the systems parameters, and calling it "highly stable" is misleading and pushes persons into thinking it's more stable than it actually is.

Recap of sub-module "state space representations"

- a set of variables is a state vector if it satisfies for that model the separation principle, i.e., the current state vector "decouples" the past with the future
- state space models are finite, and first order vectorial models

notes

- the most important remarks from this sub-module are these ones