Systems Laboratory, Spring 2025

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notes

- welcome to the course!
- on this side of this document you will find notes that accompany the text typically visualized in class
- these notes are meant to convey the messages that are not displayed in the text on the side, and basically constitute what the teacher intends to say in class

Table of Contents I

notes

- this is the table of contents of this document; each section corresponds to a specific part of the course computing free evolutions and forced responses of LTI systems

notes

Contents map

developed content units	**taxonomy levels**
free evolution	u1, e1
forced response	u1, e1

prerequisite content units	**taxonomy levels**
LTI ODE	u1, e1
convolution	u1, e1
partial fraction decomposition	u1, e1

notes

Main ILO of sub-module "computing free evolutions and forced responses of LTI systems"

Compute free evolutions and forced responses of LTI systems
using Laplace-based formulas (but only as procedural tools)

notes

- by the end of this module you shall be able to do this

Disclaimer

the formulas introduced in this module shall be taken as "ex machina"

notes

- in other words, they are given and to be assumed as true
- in other courses or modules they will be derived from other principles

Focus in this module = on ARMA models

$$y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots + b_0 u$$

with $^{(i)}$ meaning the i-th time derivative. *Discussion:* why is the LHS $y^{(n)}$ and not $a_n y^{(n)}$? *Discussion:* and which initial conditions shall we consider?

notes

- generalizing the LTIs we saw until now, we can arrive at these models, and in this module we will treat only these models (there may be other generalizations but you will see them in other modules)
- the $a_{n-1}y^{(n-1)} + \ldots + a_0 y$ part is called Auto-Regressive
- the $b_m u^{(m)} + \ldots + b_0 u$ part is called Moving-Average
- these names make more sense in discrete time systems of the type $y(k+n) = a_{n-1}y(k+n-1) + \ldots + a_0 y(k) + b_m u^(k+m) + \ldots + b_0 u(k)$ and k a discrete time index. Here we see that the a's correspond to an autoregression, and the b's to the coefficients of a moving average. In any case we use ARMA for both continuous and discrete dynamics of these types
- note that in mechanical systems like motors, the derivatives of u are meaningful because they capture the system's response to changes in the input signal, accounting for physical constraints like inertia
- this is because if we were having $a_n y^{(n)}$ on the left hand side then we could divide all the a's and b's on the right hand side and get the same dynamics
- so we prefer to work with monic polynomials (i.e., in which the leading coefficient, that is the nonzero coefficient of highest degree, is equal to 1) because we have less numbers to carry around (plus it will be convenient for other purposes that we will see later on in the course)
- as for the initial conditions that one shall consider, we typically assume all the conditions on the u equal to zero, while on the y they may be different from zero

Laplace transforms - links for who would like to get more info about them

Laplace transforms = extension of Fourier transforms; interesting material:

- `https://www.youtube.com/watch?v=r6sGWTCMz2k` (Fourier series)
- `https://www.youtube.com/watch?v=spUNpyF58BY` (Fourier transforms)
- `https://www.youtube.com/watch?v=nmgFG7PUHfo` (on the historical importance of Fast Fourier Transforms)
- `https://www.youtube.com/watch?v=7UvtU75NXTg` (Laplace Transforms, in math)
- `https://www.youtube.com/watch?v=n2y7n6jw5d0` (Laplace Transforms, graphically)

notes

- note that this module treats the formulas as "given", so who wants to look at these links shall do only for self-interest, not for preparing oneself for exercises at the exam related to this module

Main usefulness: convolution in time transforms into multiplication in Laplace-domain, and viceversa

$$\begin{cases} H(s) = \mathcal{L}\left\{h(t)\right\} \\ U(s) = \mathcal{L}\left\{u(t)\right\} \end{cases} \quad \Longrightarrow \quad \mathcal{L}\left\{h * u(t)\right\} = H(s)U(s)$$

Noticeable name: *transfer function* $\left(= H(s) = \mathcal{L}\left\{\text{impulse response}\right\}\right)$

notes

- this is by far one of the most important properties of Laplace transforms for our purposes: convolution in one of the domains will be multiplication in the other
- this implies that instead of computing $u * y$, if computing H and U is fast, and if inverting HU is fast, that way is preferrable
- the name "transfer function" is an important one and you will hear about it quite often

An intuitive explanation of the usefulness of the Laplace transform in automatic control

$$\begin{tikzcd}
\text{initial value problem} & \text{algebraic problem} \\
\boxed{\dot{x}(t) + x(t) = 0, \quad x(0) = 2} \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \boxed{sX(s) - x(0) + X(s) = 0} \arrow[d, "\text{algebra}"] \\
\boxed{x(t) = 2e^{-t}} & \boxed{X(s) = 2/(s+1)} \arrow[l, "\mathcal{L}^{-1}"] \\
\text{solution in the time domain} & \text{solution in the frequency domain}
\end{tikzcd}$$

notes

- this means that we can follow this scheme
- in other words, for complicated differential equations Laplace transform allow us to solve the problem algebraically. This is often much easier than solving the ODE directly

First set of formulas to memorize: Laplace-transforming derivatives

(these will be motivated in other courses)

$$\mathcal{L}\{\dot{x}\} = sX(s) - x(0)$$

$$\mathcal{L}\{\ddot{x}\} = s^2X(s) - sx(0) - \dot{x}(0)$$

$$\mathcal{L}\{\dddot{x}\} = s^3X(s) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0)$$

$$\mathcal{L}\{x^m\} = \ldots$$

notes

- these formulas shall be remembered by heart

Example: spring mass system

$$\ddot{y} = -\frac{f}{m}\dot{y} - \frac{k}{m}y + u$$
$$\Downarrow$$
$$s^2Y(s) - sy_0 - \dot{y}_0 = -\frac{f}{m}\Big(sY(s) - y_0\Big) - \frac{k}{m}Y(s) + U(s)$$
$$\Downarrow$$
$$s^2Y(s) + \frac{f}{m}sY(s) + \frac{k}{m}Y(s) = +sy_0 + \dot{y}_0 + \frac{f}{m}y_0 + U(s)$$
$$\Downarrow$$
$$Y(s) = \frac{y_0\left(\frac{f}{m} + s\right) + \dot{y}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} + \frac{1}{s^2 + \frac{f}{m}s + \frac{k}{m}}U(s)$$

notes

- let's then start this path building on top of previous results
- more precisely, from the fact that using Laplace transforms we were able to characterize the free evolution of second order LTI systems
- and $Y(s) \neq 0$ happens when the initial conditions of the system are not null

And what shall we do once we get this?

generalizing the previous slide: $Y(s) = \frac{M(s)}{A(s)} + \frac{B(s)}{A(s)} U(s)$

with

- $\frac{M(s)}{A(s)}$ = Laplace transform of the free evolution
- $\frac{B(s)}{A(s)} U(s)$ = Laplace transform of the forced response

$\Longrightarrow$ we shall anti-transform; how? Main 2 cases:

- either $U(s) = \frac{\text{polynomial in } s}{\text{polynomial in } s}$
- or $U(s)$ = something else

notes

- now we have this first result, where we note that the total signal is the sum of the two individual signals "free evolution" plus "forced response", but in the Laplace domain
- for the sake of this module we consider that $U(s)$ may be rational or not

Question 1

Is the Laplace transform of the signal

$$h(t) = \begin{cases} \frac{1}{t+1} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a rational Laplace transform?

Potential answers:

I: **(wrong)** yes
II: **(correct)** no
III: **(wrong)** it depends
IV: **(wrong)** I don't know

Solution 1:

A Laplace transform is rational if and only if it can be expressed as

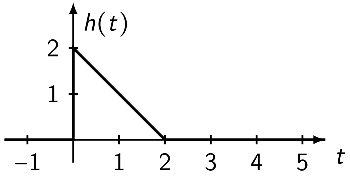
$$H(s) = \frac{N(s)}{D(s)}$$

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

Is the Laplace transform of the signal $h(t)$ below a rational Laplace transform?



Potential answers:

I: **(wrong)** yes
II: **(correct)** no
III: **(wrong)** it depends
IV: **(wrong)** I don't know

notes

- see the associated solution(s), if compiled with that ones :)

Solution 1:

A Laplace transform is rational if and only if it can be expressed as

$$H(s) = \frac{N(s)}{D(s)}$$

first case: rational $U(s)$

notes

How to do if $U(s) = \frac{\text{polynomial in } s}{\text{polynomial in } s}$

$$Y(s) = \frac{M(s)}{A(s)} + \frac{B(s)}{A(s)} U(s) \quad \mapsto \quad Y(s) = \frac{M(s)}{A(s)} + \frac{C(s)}{D(s)}$$

write each of the two parts of the signal as

$$\frac{N(s)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots}$$

notes

- in this case we have a situation for which we can write both elements as polynomial over polynomial

Next step: partial fraction decomposition

- **case single poles:** if $\frac{N(s)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots}$ is s.t. $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots$ then there exist $\alpha_1, \alpha_2, \alpha_3, \ldots$ s.t.

$$\frac{N(s)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots} = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2} + \frac{\alpha_3}{s-\lambda_3} + \cdots \qquad (1)$$

- **case repeated poles:** if some poles are repeated, then there exist $\alpha_{1,1}, \ldots, \alpha_{1,n1}, \alpha_{2,1}, \ldots, \alpha_{2,n2}, \ldots,$ s.t.

$$\frac{N(s)}{(s-\lambda_1)^{n1}(s-\lambda_2)^{n2}\cdots} = \frac{\alpha_{1,1}}{s-\lambda_1} + \ldots + \frac{\alpha_{1,n1}}{(s-\lambda_1)^{n1}} + \frac{\alpha_{2,1}}{s-\lambda_2} + \ldots + \frac{\alpha_{2,n2}}{(s-\lambda_2)^{n2}} + \ldots \qquad (2)$$

"But how do I compute α_1, α_2, etc.?" $\mapsto$
`en.wikipedia.org/wiki/Partial_fraction_decomposition`
(tip: start from `en.wikipedia.org/wiki/Heaviside_cover-up_method`)

notes

- let's remember that the partial fraction decomposition concept helps us factorizing ratios of polynomials in a sum of simpler ratios
- and let's also remember that there is the possibility of having multiple poles (something that, as we will see very soon, connects with the concept of non-trivial Jordan structure of the A expressing this LTI system)
- in case somebody does not remember how to do it, there is a couple of resources that may help re-gaining knowledge on this tool

Anti-transforming in the rational $U(s)$ case

$$\text{if } Y(s) = \frac{\alpha_{1,1}}{s-\lambda_1} + \ldots + \frac{\alpha_{1,n1}}{(s-\lambda_1)^{n1}} + \frac{\alpha_{2,1}}{s-\lambda_2} + \ldots + \frac{\alpha_{2,n2}}{(s-\lambda_2)^{n2}} + \ldots \text{ then use}$$

$$\mathcal{L}\left\{t^n e^{\lambda t}\right\} = \frac{n!}{(s-\lambda)^{n+1}} \quad \leftrightarrow \quad \mathcal{L}^{-1}\left\{\frac{n!}{(s-\lambda)^{n+1}}\right\} = t^n e^{\lambda t}$$

notes

- given this transform, $y(t)$ is then immediately a sum of terms of the type $t^n e^{\lambda t}$ for opportune n's that depend on the specific λ
- we see that this must connect with the structure of the Jordan form of the A expressing this LTI
- we will reinforce this connection later on – the important for now is to realize that it exists
- now either all the terms are simple, or there are some repeated lambda's

Numerical Example: Inverse Laplace Transform of a Rational Function

$$Y(s) = \frac{3}{s-2} + \frac{4}{(s-2)^2} + \frac{5}{s+1}$$

goal $=$ compute the inverse Laplace transform $y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\}$

notes

- now let's do this exercise. Let's assume we started from an opportune $u(t)$ and ARMA model and initial conditions such that the general formula

$$Y(s) = \frac{M(s)}{A(s)} + \frac{B(s)}{A(s)}U(s)$$

has brought us to this specific $Y(s)$

Step 1: Identify the terms

$$Y(s) = \frac{3}{s-2} + \frac{4}{(s-2)^2} + \frac{5}{s+1}$$

Here:

- $\lambda_1 = 2$, with coefficients $\alpha_{1,1} = 3$ and $\alpha_{1,2} = 4$
- $\lambda_2 = -1$, with coefficient $\alpha_{2,1} = 5$

notes

- we get immediately this, by applying what we saw in the previous slides

Step 2: Apply the inverse Laplace transform formula

by means of

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s-\lambda)^{n+1}}\right\} = t^n e^{\lambda t}$$

we compute the inverse Laplace transform of each term:

- $\mathcal{L}^{-1}\left\{\frac{3}{s-2}\right\} = 3e^{2t}$
- $\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2}\right\} = 4te^{2t}$
- $\mathcal{L}^{-1}\left\{\frac{5}{s+1}\right\} = 5e^{-t}$

notes

- then we get this

Step 3: Combine the results

then we have that the inverse Laplace transform $y(t)$ is the sum of the individual transforms, i.e.,

$$y(t) = 3e^{2t} + 4te^{2t} + 5e^{-t}$$

notes

- then we get this

Another Example: Inverse Laplace Transform with Complex Conjugate Terms

let

$$Y(s) = \frac{2s+3}{s^2+2s+5}$$

and the goal to be to compute the inverse Laplace transform $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

notes

- assume this situation

Step 1: Factor the denominator

note: $s^2 + 2s + 5$ has complex conjugate roots, indeed

$$s^2 + 2s + 5 = (s+1)^2 + 4$$

and thus

$$Y(s) = \frac{2s+3}{(s+1)^2 + 4}$$

notes

- note that we have complex conjugate pairs

Step 2: Express in terms of standard forms

rewrite $Y(s)$ to match the standard forms for inverse Laplace transforms involving complex conjugates, i.e.,

$$Y(s) = \frac{2(s+1)+1}{(s+1)^2+4} = 2 \cdot \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}.$$

notes

- note that we have complex conjugate pairs

Step 3: Apply the inverse Laplace transform formula

since

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos(bt),$$

$$\mathcal{L}^{-1}\left\{\frac{b}{(s+a)^2+b^2}\right\} = e^{-at}\sin(bt),$$

we have, for the various terms:

- $\mathcal{L}^{-1}\left\{2\cdot\frac{s+1}{(s+1)^2+4}\right\} = 2e^{-t}\cos(2t)$
- $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = \frac{1}{2}e^{-t}\sin(2t)$

notes

- and here we are just using formulas

Step 4: Combine the results

$$y(t) = 2e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)$$

notes

- and here we are just using formulas

Extremely important result

a LTI in free evolution behaves as a combination of terms $e^{\lambda t}$, $te^{\lambda t}$, $t^2 e^{\lambda t}$, etc. for a set of different λ's and powers of t, called the *modes* of the system

Discussion: assuming that we have two modes, $e^{-0.3t}$ and $e^{-1.6t}$, so that

$$y(t) = \alpha_1 e^{-0.3t} + \alpha_2 e^{-1.6t}.$$

What determines α_1 and α_2?

notes

- these signals are thus somehow describing the natural way a free evolution evolves
- we already saw them with Jordan forms, and we did not give them a name then
- but they have a specific name: they are the modes of a LTI
- these numbers are given by the initial conditions of the system

second case: irrational $U(s)$

notes

In this case we cannot use partial fractions decompositions as before

from $Y(s) = \dfrac{M(s)}{A(s)} + \dfrac{B(s)}{A(s)} U(s)$ we follow the algorithm

- find $y_{\mathrm{free}}(t)$ from PFDs of $\dfrac{M(s)}{A(s)}$ as before
- find the impulse response $h(t)$ from PFDs of $\dfrac{B(s)}{A(s)}$ as before
- find $y_{\mathrm{forced}}(t)$ as $h * u(t)$

notes

- in this case we need to find the various terms independently

Summarizing

Compute free evolutions and forced responses of LTI systems using Laplace-based formulas (but only as procedural tools)

- Laplace the ARMA
- if $u(t)$ admits a rational $U(s)$ then write $Y(s) = \dfrac{\text{polynomial}}{\text{polynomial}}$, do PFD, and do inverse-Laplaces
- if $u(t)$ does not admit a rational $U(s)$, do similarly as before but do PFD only for the free evolution and impulse response, and find the forced response by means of convolution

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

Two essential libraries

- `https://python-control.readthedocs.io/en/0.10.1/generated/control.modal_form.html`
- `https://docs.sympy.org/latest/modules/physics/control/lti.html`

notes

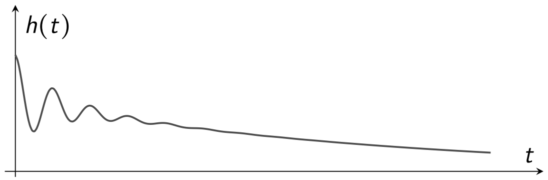
- these libraries provide you the necessary tools to perform modal analysis as here

Self-assessment material

notes

Question 3

Which type of LTI system may produce the impulse response $h(t)$ represented in the picture?



Potential answers:

I: **(wrong)** first order
II: **(wrong)** second order
III: **(correct)** at least third order
IV: **(wrong)** I do not know

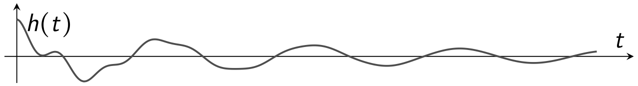
Solution 1:

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

Which type of LTI system may produce the impulse response $h(t)$ represented in the picture?



Potential answers:

I: **(wrong)** first order
II: **(wrong)** second order
III: **(wrong)** third order
IV: **(correct)** at least fourth order
V: **(wrong)** I do not know

Solution 1:

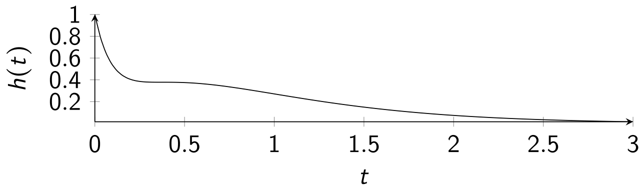
The impulse response may be decomposed as a sum of two decaying oscillatory behaviors, i.e., as $h(t) = e^{\alpha t} \cos(\omega_1 t) + e^{\beta t} \cos(\omega_2 t)$, as in the figure below. The first part $e^{\alpha t} \cos(\omega_1 t)$ decays faster than the second part $e^{\beta t} \cos(\omega_2 t)$, and is also associated to a cosine oscillating at a higher frequency than the second. Hence this impulse response associates with two modes, both relating to a second order

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

Which type of LTI system may produce the impulse response $h(t)$ below?



Potential answers:

I: **(wrong)** first order
II: **(wrong)** second order
III: **(correct)** at least third order
IV: **(wrong)** I do not know

indicate a transfer function with two complex conjugates stable poles on the left

Solution 1:

Looking at the graph of $h(t)$, we decompose it in the sum of two different modes:

notes

- see the associated solution(s), if compiled with that ones :)

Question 6

What is the primary purpose of using Laplace transforms in solving LTI systems?

Potential answers:

I: **(wrong)** To convert differential equations into algebraic equations for easier solving.
II: **(correct)** To transform convolution in the time domain into multiplication in the Laplace domain.
III: **(wrong)** To directly compute the eigenvalues of the system matrix.
IV: **(wrong)** To eliminate the need for initial conditions in solving differential equations.
V: **(wrong)** I do not know.

Solution 1:

The primary purpose of using Laplace transforms is to transform convolution in the time domain into multiplication in the Laplace domain, simplifying the solution of differential equations.

notes

- see the associated solution(s), if compiled with that ones :)

Question 7

What is the correct form of the inverse Laplace transform of $\frac{1}{(s-\lambda)^2}$?

Potential answers:

I: **(wrong)** $e^{\lambda t}$
II: **(wrong)** $te^{\lambda t}$
III: **(correct)** $te^{\lambda t}$
IV: **(wrong)** $\frac{1}{2}t^2e^{\lambda t}$
V: **(wrong)** I do not know.

Solution 1:

The inverse Laplace transform of $\frac{1}{(s-\lambda)^2}$ is $te^{\lambda t}$.

notes

- see the associated solution(s), if compiled with that ones :)

Question 8

What is the inverse Laplace transform of $\dfrac{s+1}{(s+1)^2+4}$?

Potential answers:

I: **(wrong)** $e^{-t}\sin(2t)$
II: **(correct)** $e^{-t}\cos(2t)$
III: **(wrong)** $e^{-t}\cos(t)$
IV: **(wrong)** $e^{-t}\sin(t)$
V: **(wrong)** I do not know.

Solution 1:

The inverse Laplace transform of $\dfrac{s+1}{(s+1)^2+4}$ is $e^{-t}\cos(2t)$.

notes

- see the associated solution(s), if compiled with that ones :)

Question 9

In the ARMA model $y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots + b_0 u$, why is the leading coefficient of $y^{(n)}$ typically set to 1?

Potential answers:

I: **(wrong)** To ensure the system is stable.
II: **(wrong)** To simplify the computation of eigenvalues.
III: **(correct)** To reduce the number of parameters and work with monic polynomials.
IV: **(wrong)** To make the system linear time-invariant.
V: **(wrong)** I do not know.

Solution 1:

The leading coefficient of $y^{(n)}$ is typically set to 1 to reduce the number of parameters and work with monic polynomials, which simplifies calculations.

notes

- see the associated solution(s), if compiled with that ones :)

Question 10

What determines the coefficients α_1 and α_2 in the free evolution response $y(t) = \alpha_1 e^{-0.3t} + \alpha_2 e^{-1.6t}$?

Potential answers:

I: **(wrong)** The eigenvalues of the system matrix.
II: **(wrong)** The input signal $u(t)$.
III: **(correct)** The initial conditions of the system.
IV: **(wrong)** The poles of the transfer function.
V: **(wrong)** I do not know.

Solution 1:

The coefficients α_1 and α_2 are determined by the initial conditions of the system.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module "computing free evolutions and forced responses of LTI systems"

- finding such signals require knowing a couple of formulas by heart
- partial fraction decomposition is king here, one needs to know how to do that

notes

- the most important remarks from this sub-module are these ones