Systems Laboratory, Spring 2025

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notes

- welcome to the course!
- on this side of this document you will find notes that accompany the text typically visualized in class
- these notes are meant to convey the messages that are not displayed in the text on the side, and basically constitute what the teacher intends to say in class

Table of Contents I

- 1D convolution in continuous time
  - Most important python code for this sub-module
  - Self-assessment material

notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

1D convolution in continuous time

notes

Contents map

developed content units	taxonomy levels
convolution	u1, e1

prerequisite content units	taxonomy levels
signal	u1, e1

notes

Main ILO of sub-module "1D convolution in continuous time"

Compute the convolution between two single dimensional continuous time signals

notes

- by the end of this module you shall be able to do this

Why convolution?

because for a LTI system with impulse response $h(t)$
it follows that $y_{\text{forced}}(t) = u * h(t)$

notes

- and indeed formally we get this result, that we do not prove for now extremely important result for LTI systems:

$$y_{\text{forced}}(t) = h * u(t) = u * h(t) :=$$

$$:= \int_{-\infty}^{+\infty} u(\tau) h(t-\tau) d\tau = \int_{-\infty}^{+\infty} h(\tau) u(t-\tau) d\tau$$

. . . and this module = what that formula actually means from graphical perspectives

notes

Additional material

Videos:

- https://www.youtube.com/watch?v=KuXjwB4LzSA
- https://www.youtube.com/watch?v=acAw5WGtzuk
- https://www.youtube.com/watch?v=IaSGqQa5O-M (for connections with probability)
- https://www.youtube.com/playlist?list=PL4iThgVpN7hmbIhHnCa7SDOOgLMoNwED_
- https://www.youtube.com/playlist?list=PL4mJLdGEHNvhCuPXsKFrnD7AaQBlMEB6a

Animations:

- https://lpsa.swarthmore.edu/Convolution/CI.html
- https://phiresky.github.io/convolution-demo/

notes

- the world is full of material on convolution - check also this stuff and not only what we do in class

Towards decomposing this formula in pieces

$$y_{\text{forced}}(t) = h * u(t) = \int_{-\infty}^{+\infty} u(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

better focus on

$$\int_{-\infty}^{+\infty} u(\tau)h(t-\tau)d\tau$$

or on

$$\int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau \ ?$$

in automatic control typically better the second

notes

- in the first option one has to "flip" u
- in the second option one has to "flip" h
- the second gives us more intuitions about the system, makes the impulse response directly interpretable
- they though give the same results

Towards decomposing this formula in pieces, small change of notation

$$y_{\text{forced}}(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau \quad \mapsto \quad y_{\text{forced}}(\text{now}) = \int_{-\infty}^{+\infty} h(\tau)u(\text{now}-\tau)d\tau$$

notes

- better doing this change of notation, so to avoid some confusion

Decomposing this formula in pieces

$$y_{\text{forced}}(\text{now}) = \int_{-\infty}^{+\infty} h(\tau)u(\text{now}-\tau)d\tau$$

$\implies$ constituent pieces =

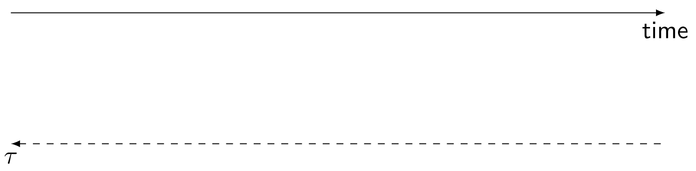
- $u(\text{now}-\tau)$
- $h(\tau)$
- $u(\text{now}-\tau)h(\tau)$
- $\int u(\text{now}-\tau)h(\tau)d\tau$

notes

- the main pieces / steps of computing a convolution are these ones

Visualizing the various pieces

$u(\text{now}-\tau)$ $\quad h(\tau)$ $\quad u(\text{now}-\tau)h(\tau)$ $\quad \int_{-\infty}^{+\infty} h(\tau)u(\text{now}-\tau)d\tau$

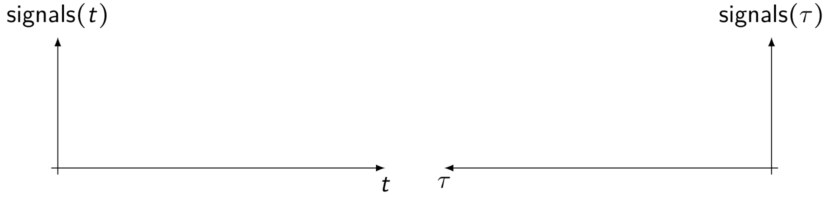


notes

- likely better watch the video in class for this animation, or try to refer to `https://lpsa.swarthmore.edu/Convolution/CI.html`

Example

$$u(t) = \begin{cases} 1 & \text{for } t \in [1,2] \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 2 & \text{for } t \in [0,1) \\ 1 & \text{for } t \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$



notes

- see the video of the lesson for the example
- note though that with the second option the meaning of h is "how important the past is to create the current output"

Another Example:

$$u(t) = \begin{cases} 1 & \text{for } t \in [1,2] \text{ and } t \in [2,3] \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 2 & \text{for } t \in [0,2) \\ 0 & \text{otherwise} \end{cases} \qquad ?$$

notes

- see the video of the lesson

Paramount message

h in $y_{\text{forced}}(t) = \int_{-\infty}^{+\infty} h(\tau) u(t-\tau) d\tau$ represents how much the past u's contribute to the current y_{forced}:

notes

- this is a very important interpretation, thus better to fix it
- for example, $h(7)$ says how much $u(t-7)$ will enter into $y(t)$ for any t
- so somehow $h(\tau)$ has an important graphical meaning
- if $h(\tau)$ decays fast, then somehow the system "forgets" the past inputs fast, and viceversa

Refreshing what we are doing and why

Dynamics of a cart: $\dot{v}(t) = -\frac{k}{m} v(t) + \frac{k}{m} F(t)$ with:

- **control input:** $u(t)$ (actuation from the motor, in this case $= F(t)$)
- **system output:** $y(t)$ (cart velocity, in this case $= v(t)$)
- **impulse response:** $h(t)$ (output corresponding to the input $\delta(t)$ assuming $y(0) = 0$)
- **free evolution:** $y_{\text{free}}(t)$ (output in time corresponding to no input, i.e., $u(t) = 0$, and initial condition $y(0)$ whatever it is)
- **forced response:** $y_{\text{forced}}(t) = u * h(t)$ (output in time corresponding to null initial condition, i.e., $y(0) = 0$, and input $u(t)$ whatever it is)
- **total response:** $y(t) = y_{\text{free}}(t) + y_{\text{forced}}(t)$

notes

- just to be sure: what we are doing here is essential
- we want to understand how to map the input into the output
- if we want to do model predictive control indeed we need to know what a certain input is going to cause to the output

Quiz time!

$$h * u(t) := \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$$

- is $h * u(t) = u * h(t)$?
- is $(\alpha h_1 + \beta h_2) * u(t) = \alpha\left(h_1 * u(t)\right) + \beta\left(h_2 * u(t)\right)$?
- if both $h(\tau) = 0$ and $u(t) = 0$ if $t < 0$, how can we simplify $y(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$?

notes

-
- obviously yes, since we can do the change of variables
-
- yes because the integrals are linear. This implies that the convolution operator is linear
-
- yes, it becomes $y(t) = \int_0^t h(\tau)u(t-\tau)d\tau$

Summarizing

Compute the convolution between two single dimensional continuous time signals

- take one of the two signals
- translate it to the "current t"
- flip it
- multiply the two signals in a pointwise fashion
- compute the integral of the result

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

Methods implementing (discrete) convolutions

- https://numpy.org/doc/2.1/reference/generated/numpy.convolve.html
- https://docs.scipy.org/doc/scipy/reference/generated/scipy.signal.convolve.html

notes

- note that these methods take discrete arrays, also multidimensional

Self-assessment material

notes

Question 1

What does the convolution integral $y_{\text{forced}}(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$ represent in the context of LTI systems?

Potential answers:

I: **(wrong)** The free evolution of the system output.
II: **(correct)** The forced response of the system output due to the input $u(t)$.
III: **(wrong)** The total response of the system, including initial conditions.
IV: **(wrong)** The impulse response of the system.
V: **(wrong)** I do not know.

Solution 1:

The convolution integral $y_{\text{forced}}(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$ represents the forced response of the system output due to the input $u(t)$. It describes how the system responds to the input when initial conditions are zero.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

Which of the following is true about the convolution operation $h * u(t)$?

Potential answers:

I: **(wrong)** It is only defined for periodic signals.
II: **(wrong)** It is only applicable to discrete-time systems.
III: **(correct)** It is commutative, i.e., $h * u(t) = u * h(t)$.
IV: **(wrong)** It requires both signals to be symmetric.
V: **(wrong)** I do not know.

Solution 1:

The convolution operation is commutative, meaning $h * u(t) = u * h(t)$. This property holds for continuous-time signals in LTI systems.

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

What does the impulse response $h(t)$ of an LTI system represent?

Potential answers:

I: **(wrong)** The input signal $u(t)$ applied to the system.
II: **(wrong)** The free evolution of the system output.
III: **(wrong)** The total response of the system, including initial conditions.
IV: **(correct)** The output of the system when the input is a Dirac delta function $\delta(t)$.
V: **(wrong)** I do not know.

Solution 1:

The impulse response $h(t)$ represents the output of the system when the input is a Dirac delta function $\delta(t)$. It characterizes the system's behavior.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

If $h(\tau) = 0$ for $\tau < 0$ and $u(t) = 0$ for $t < 0$, how can the convolution integral $y(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$ be simplified?

Potential answers:

I: **(correct)** $y(t) = \int_0^t h(\tau)u(t-\tau)d\tau$
II: **(wrong)** $y(t) = \int_0^{+\infty} h(\tau)u(t-\tau)d\tau$
III: **(wrong)** $y(t) = \int_{-\infty}^{+\infty} h(\tau)u(\tau)d\tau$
IV: **(wrong)** $y(t) = \int_{-\infty}^{0} h(\tau)u(t-\tau)d\tau$
V: **(wrong)** I do not know.

Solution 1:

If $h(\tau) = 0$ for $\tau < 0$ and $u(t) = 0$ for $t < 0$, the convolution integral simplifies to $y(t) = \int_0^t h(\tau)u(t-\tau)d\tau$, as the integrand is zero outside this interval.

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

What is the graphical interpretation of $h(\tau)$ in the convolution integral $y_{\text{forced}}(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau$?

Potential answers:

I: **(wrong)** It represents the future inputs of the system.
II: **(correct)** It represents how much past inputs contribute to the current output.
III: **(wrong)** It represents the free evolution of the system.
IV: **(wrong)** It represents the total energy of the system.
V: **(wrong)** I do not know.

Solution 1:

The term $h(\tau)$ in the convolution integral represents how much past inputs $u(t-\tau)$ contribute to the current output $y(t)$. This is a key interpretation in LTI systems.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module "1D convolution in continuous time"

- convolution is an essential operator, since it can be used for LTI systems to compute forced responses
- its graphical interpretation aids interpreting impulse responses as how the past inputs contribute to current outputs

notes

- the most important remarks from this sub-module are these ones