Systems Laboratory, Spring 2025

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notes

- welcome to the course!
- on this side of this document you will find notes that accompany the text typically visualized in class
- these notes are meant to convey the messages that are not displayed in the text on the side, and basically constitute what the teacher intends to say in class

Table of Contents I

- what is the superposition principle, and what does it imply
  - Most important python code for this sub-module
  - Self-assessment material

notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

what is the superposition principle, and what does it imply

notes

Contents map

developed content units	taxonomy levels
superposition principle	u1, e1

prerequisite content units	taxonomy levels
LTI ODE	u1, e1

notes

Main ILO of sub-module

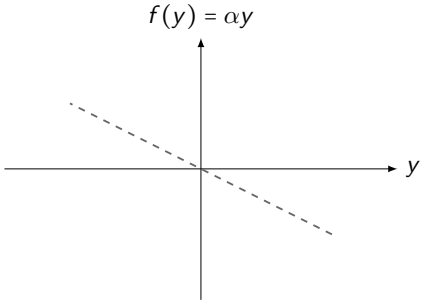
"what is the superposition principle, and what does it imply"

Describe the importance of the superposition principle to analyze LTI systems

notes

- by the end of this module you shall be able to do this

Starting with graphs



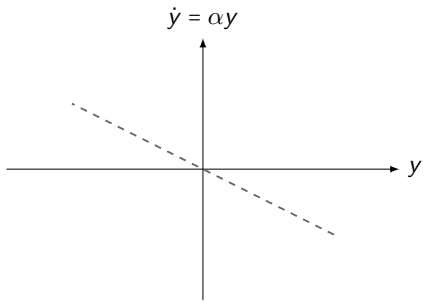
implications/definition of linearity:

- $f(x+y) = f(x) + f(y)$
- $f(\alpha y) = \alpha f(y)$

notes

- looking at this graph, we note that these two properties hold
- and this holds only because of linearity. If we are having an affine map, for example, the second would not hold

What if we interpret this as an ODE?



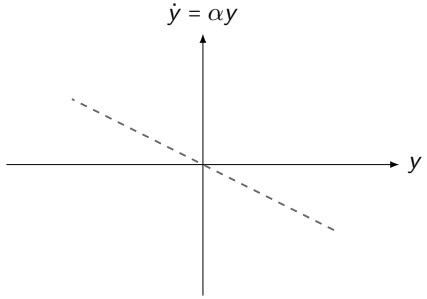
$\Longrightarrow$ an LTI system, for which

$$\dot{y} = \alpha y \quad \text{is solved by} \quad y(t) = y(0)\exp(\alpha t) \quad \forall y(0), \alpha, t$$

notes

- you may verify this solution by doing the direct verification

And can we build on top of this?



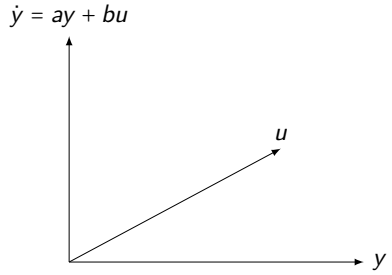
- $y'(0) = 2 \mapsto y'(t) = 2\exp(\alpha t)$
- $y''(0) = 3 \mapsto y''(t) = 3\exp(\alpha t)$
- $y'''(0) = 3 + 2 \mapsto y'''(t) = (3 + 2)\exp(\alpha t)$

$$y'(0) \; + \; y''(0) \quad \mapsto \quad y'(t) \; + \; y''(t)$$

notes

- looking at what we found in the previous module, this holds because the solutions to linear ODEs are exponentials passing by the initial conditions and whose exponent is always *a*
- and this holds only because of linearity

Further generalization



- $\{y'(0), u'\} \mapsto y'(t)$
- $\{y''(0), u''\} \mapsto y''(t)$
- $\{y'(0) + y''(0), u' + u''\} \mapsto y'(t) + y''(t)$

notes

- we can then generalize the previous result in this way, when we have linearity

Aiding intuitions with math

Linearity implies that if $\{y', u', y'(0)\}$ and $\{y'', u'', y''(0)\}$ satisfy

$$\begin{cases} \dfrac{dy'(t)}{dt} & = ay'(t) + bu'(t) \\ y'(0) & = y_0' \\ \dfrac{dy''(t)}{dt} & = ay''(t) + bu''(t) \\ y''(0) & = y_0'' \end{cases} \tag{1}$$

then their sum also satisfies

$$\begin{cases} \dfrac{d\big(\alpha' y'(t) + \alpha'' y''(t)\big)}{dt} & = a\big(\alpha' y'(t) + \alpha'' y''(t)\big) + b\big(\alpha' u'(t) + \alpha'' u''(t)\big) \\ \alpha' y'(0) + \alpha'' y''(0) & = \alpha' y_0' + \alpha'' y_0'' \end{cases} \tag{2}$$

notes

- let's start realizing that this holds because derivatives hold

Rephrasing

Linearity implies that if $\{y', u', y'(0)\}$ and $\{y'', u'', y''(0)\}$ satisfy the ODE then also their sum $\{y' + y'', u' + u'', y'(0) + y''(0)\}$ satisfies the ODE.

The superposition principle in words

in LTI systems
combining inputs and initial conditions
produces a total effect
that is the linear combination
of that effects
one would get with the individual causes
each acting separately

notes

- this is the same thing in the previous slide, written in words
- then the previous math basically says this

Important: the superposition principle works with any LTI

Will be repeated and stated again precisely later on

the proof holds for every system that generalizes $\dot{y} = ay + bu$,
i.e., every "linear combination of dots of y = linear combination of dots of u"

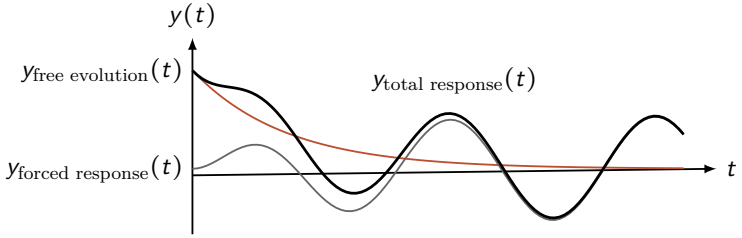
notes

- we have moreover this generalization, that we will see again and again, since derivatives are linear

Superposition principle $\Longrightarrow$ response of LTIs = free evolution + forced response

assume:

- $\dot{y} = ay + bu$
- $\{u(t) = 0(t), \quad y(0) \neq 0\}$ causes $y_{\text{free evolution}}(t)$
- $\{u(t) \neq 0(t), \quad y(0) = 0\}$ causes $y_{\text{forced response}}(t)$



then $\{u(t) \neq 0(t), y(0) \neq 0\}$ causes $y_{\text{free evolution}}(t) + y_{\text{forced response}}(t)$

notes

- the generalization in the previous slide can immediately be useful to show something very important, i.e., that we can decompose the general solution in to main parts, as here

A mnemonic scheme

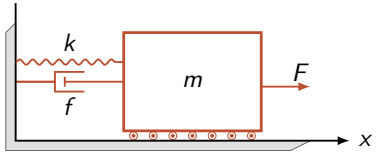
(only for LTI systems!!)

$$(u, y_0) = (0, y_0) + (u, 0)$$

total response = free evolution + forced response

notes

Continuing with some intuitions

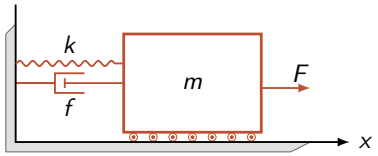


Discussion: how will the cart move if I use $u(t) = \sin(\omega t)$ starting from a resting state? (only intuitively, assuming everything ideal)
And what about if $u(t) = 2\sin(\omega t)$?
And what about $u(t) = \sin(\omega' t) + \sin(\omega'' t)$?
And what about $u(t) = \alpha' \sin(\omega' t) + \alpha'' \sin(\omega'' t)$?

notes

- we will see later on precisely; for now let's say that it moves somehow
- here intuition may not help: anyway it makes the same movement as before but with twice the amplitude
- here intuition may again not help: it makes the sum of the two movements
- here intuition may again not help: it makes the sum of the two movements scaled

Refining the intuitions



Assume to have measured

$$\boldsymbol{y}'(0), u'(t) \mapsto y'(t) \qquad \boldsymbol{y}''(0), u''(t) \mapsto y''(t)$$

Saying "this system is linear" means assuming $\forall \alpha', \alpha'' \in \mathbb{R}$

$$\alpha' \boldsymbol{y}'(0) \alpha' \boldsymbol{y}''(0), u'(t) \mapsto y'(t) \qquad \boldsymbol{y}''(0), u''(t) \mapsto y''(t)$$

thus assuming that from a resting state the input $u(t) = \alpha \sin(\omega_\alpha t) + \beta \sin(\omega_\beta t)$ causes $y(t) = \alpha y_{\omega\alpha}(t) + \beta y_{\omega\beta}(t)$

notes

- so this is the solution to the previous question

Summarizing

Describe the importance of the superposition principle to analyze LTI systems

- it makes us able to say "total = free + forced"

notes

- you should now be able to do this, following the pseudo-algorithm in the itemized list

Most important python code for this sub-module

notes

Suggestion

part of the SciPy library (`scipy.signal`) provides tools for working with LTI systems, including creating transfer functions, state-space representations, and analyzing system responses (stuff that will be seen in the next modules)

notes

- check this library, you will use it

Self-assessment material

notes

Question 1

What does the superposition principle imply for LTI systems?

Potential answers:

I: **(wrong)** The total response is the product of the free evolution and forced response.
II: **(correct)** The total response is the sum of the free evolution and forced response.
III: **(wrong)** The total response is independent of the initial conditions.
IV: **(wrong)** The total response is only determined by the input.
V: **(wrong)** I do not know.

Solution 1:

The superposition principle implies that the total response of an LTI system is the sum of the free evolution (response due to initial conditions) and the forced response (response due to the input). This is a fundamental property of linear systems.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

Which of the following is a necessary condition for the superposition principle to hold in a system?

Potential answers:

I: **(wrong)** The system must be nonlinear.
II: **(correct)** The system must be linear and time-invariant.
III: **(wrong)** The system must have time-varying parameters.
IV: **(wrong)** The system must be unstable.
V: **(wrong)** I do not know.

Solution 1:

The superposition principle holds only for Linear Time-Invariant (LTI) systems. Nonlinear or time-varying systems do not satisfy the superposition principle.

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

What is the free evolution of an LTI system?

Potential answers:

I: **(wrong)** The response of the system to a nonzero input with zero initial conditions.
II: **(correct)** The response of the system to zero input with nonzero initial conditions.
III: **(wrong)** The steady-state response of the system.
IV: **(wrong)** The transient response of the system.
V: **(wrong)** I do not know.

Solution 1:

The free evolution of an LTI system is the response of the system when the input is zero, but the initial conditions are nonzero. It represents the system's natural behavior without external forcing.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

If an LTI system has an input $u(t) = \alpha' u'(t) + \alpha'' u''(t)$ and initial conditions $y(0) = \alpha' y'(0) + \alpha'' y''(0)$, what is the total response $y(t)$?

Potential answers:

I: **(wrong)** $y(t) = \alpha' y'(t) \cdot \alpha'' y''(t)$
II: **(correct)** $y(t) = \alpha' y'(t) + \alpha'' y''(t)$
III: **(wrong)** $y(t) = \alpha' y'(t) - \alpha'' y''(t)$
IV: **(wrong)** $y(t) = \alpha' y'(t) / \alpha'' y''(t)$
V: **(wrong)** I do not know.

Solution 1:

For an LTI system, the total response $y(t)$ is the linear combination of the individual responses $y'(t)$ and $y''(t)$, scaled by α' and α'', respectively. This is a direct consequence of the superposition principle.

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

What is the forced response of an LTI system?

Potential answers:

I: **(correct)** The response of the system to a nonzero input with zero initial conditions.

II: **(wrong)** The response of the system to zero input with nonzero initial conditions.

III: **(wrong)** The response of the system to a step input.

IV: **(wrong)** The response of the system to a sinusoidal input.

V: **(wrong)** I do not know.

Solution 1:

The forced response of an LTI system is the response of the system when the input is nonzero, but the initial conditions are zero. It represents the system's behavior due to external forcing.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module

"what is the superposition principle, and what does it imply"

- superposition principle helps logically separating specific causes into specific effects
- linear ODEs $\Longrightarrow$ superposition principle
- superposition principle $\Longrightarrow$ "whole = free + forced"
- nonlinear systems WON'T satisfy this principle!

notes

- the most important remarks from this sub-module are these ones