# Systems Laboratory, Spring 2025

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notes

- welcome to the course!
- on this side of this document you will find notes that accompany the text typically visualized in class
- these notes are meant to convey the messages that are not displayed in the text on the side, and basically constitute what the teacher intends to say in class

## Table of Contents I





notes

- this is the table of contents of this document; each section corresponds to a specific part of the course

# building and interpreting phase portraits



notes

## Contents map

| developed content units | taxonomy levels |
|---|---|
| phase portrait | u1, e1 |

| prerequisite content units | taxonomy levels |
|---|---|
| ODE | u1, e1 |



notes

## Main ILO of sub-module "building and interpreting phase portraits"

**Construct** and interpret phase portraits of first- and second-order autonomous ODEs using qualitative analysis techniques

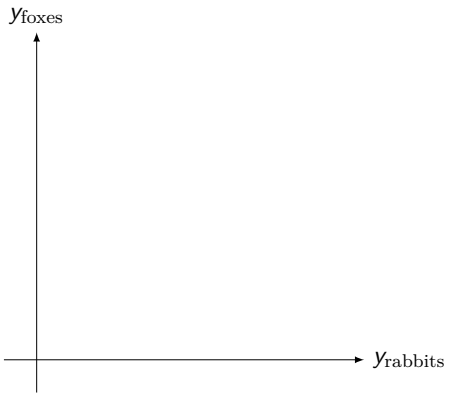


notes

- by the end of this module you shall be able to do this

## Starting with an example: a Lotka-Volterra model (≠ real world):

$$\begin{cases} \dot{y}_{\text{rabbits}} &= 0.4 \cdot y_{\text{rabbits}} - 0.5 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \\ \dot{y}_{\text{foxes}} &= -3 \cdot y_{\text{foxes}} + 0.7 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \end{cases}$$



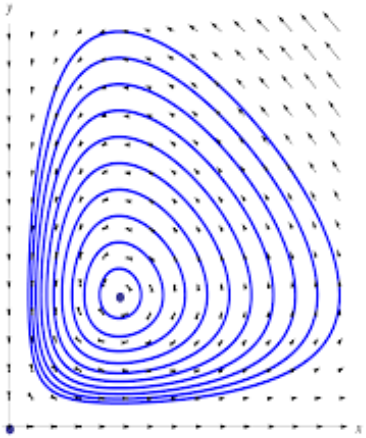



notes

- with this example we build things graphically, best to watch the video of the module

## The result, if we were plotting everything

$$\begin{cases} \dot{y}_{\text{rabbits}} &= 0.4 \cdot y_{\text{rabbits}} - 0.5 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \\ \dot{y}_{\text{foxes}} &= -3 \cdot y_{\text{foxes}} + 0.7 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \end{cases}$$

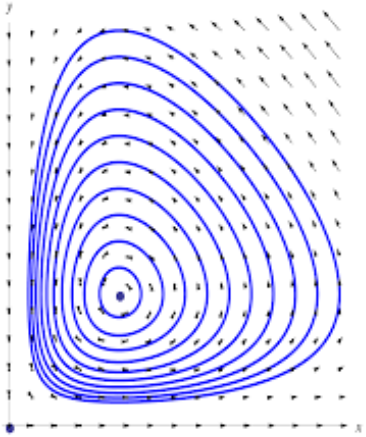




notes

- if we were doing things graphically by hand we would eventually get this

## Phase Portrait = a graphical representation of the trajectories of a dynamical system in the state space

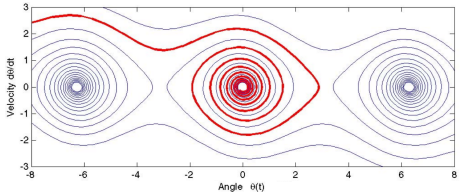




notes

- A phase portrait provides insight into the qualitative behavior of a system without requiring explicit solutions.
- It helps visualize equilibria, stability, and the general flow of solutions in state space.

## Which system is this one?

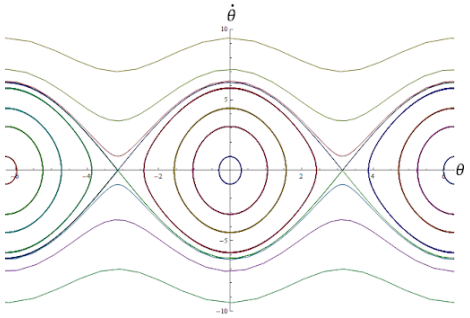



notes

- pendulum with friction



## And this one?




notes

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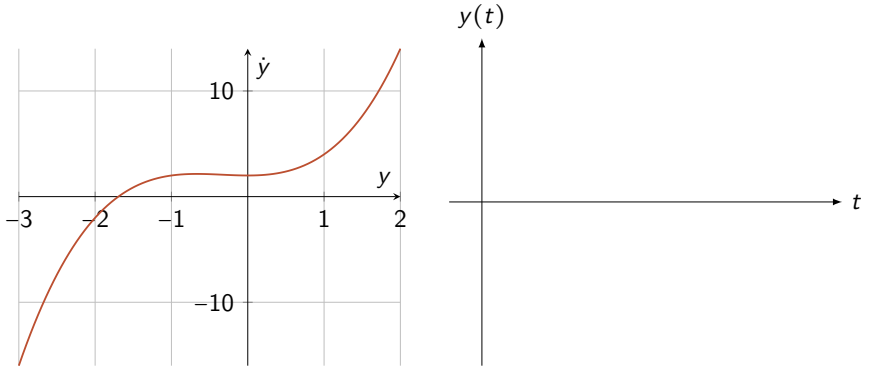
## Phase Portraits for first-order ODEs



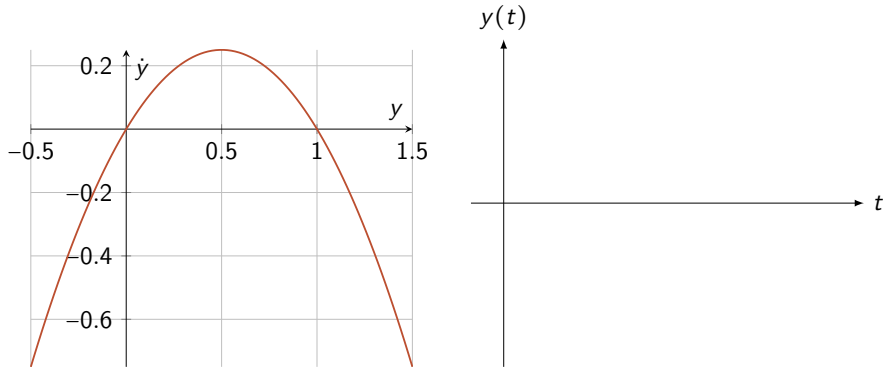

drawing the phase portrait as a 2-D thing in this case is a big error



notes

- The phase portrait consists of a one-dimensional state space (the $y$-axis).
- Equilibrium points are found by setting $f(y) = 0$.
- The sign of $f(y)$ determines the direction of motion.

## Another example: $\dot{y} = y(1 - y)$



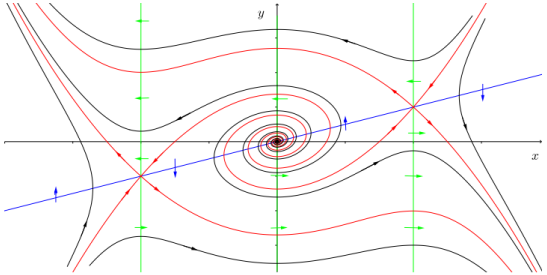

- equilibria: $y = 0$ and $y = 1$
- if $y < 0$ or $y > 1$, $\dot{y}$ is negative (flow left)
- if $0 < y < 1$, $\dot{y}$ is positive (flow right)



notes

- The stability of equilibrium points can be determined by the sign of $f(x)$.
- $x = 0$ is unstable, $x = 1$ is stable.

## Interpreting Phase Portraits



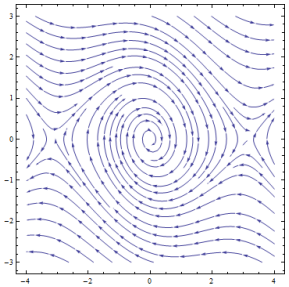

- equilibria: where trajectories do not move
- limit cycles: closed trajectories indicating periodic behavior



notes

- Understanding phase portraits helps predict long-term system behavior.
- Nonlinear systems may exhibit complex structures like attractors or chaotic dynamics.

## Interpreting Phase Portraits



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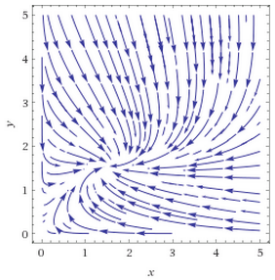


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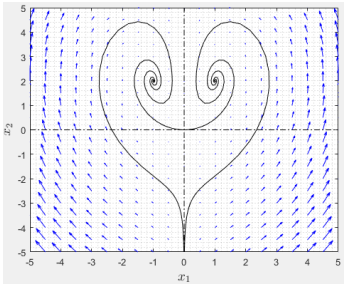


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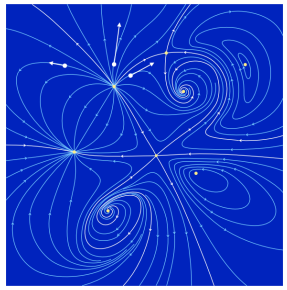

- equilibria: where trajectories do not move
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notes

- Understanding phase portraits helps predict long-term system behavior.
- Nonlinear systems may exhibit complex structures like attractors or chaotic dynamics.

Most important python code for this sub-module



notes

## Tutorial on how to plot phase portraits

`https://aleksandarhaber.com/phase-portraits-of-state-space-models-and-differential-equations-in-python/`



notes

- that tutorial is very well made, check it

Self-assessment material



notes

## Question 1

What is the primary purpose of a phase portrait?

**Potential answers:**

I: **(wrong)** To find the exact numerical solution of a system
II: **(correct)** To visualize the qualitative behavior of a dynamical system
III: **(wrong)** To approximate the integral of a function
IV: **(wrong)** To determine the frequency response of a system
V: **(wrong)** I do not know

**Solution 1:**

A phase portrait is a graphical representation of the trajectories of a system in state space, giving insight into equilibrium points, stability, and system behavior without solving the equations explicitly.

- building and interpreting phase portraits 2

notes

- see the associated solution(s), if compiled with that ones :)

## Question 2

How do you determine equilibrium points in a phase portrait of a first-order system $\dot{y} = f(y)$?

**Potential answers:**

I: **(wrong)** By solving $\dot{y} = 0$ for all values of $t$
II: **(correct)** By solving $f(y) = 0$ for $y$
III: **(wrong)** By integrating $f(y)$ over time
IV: **(wrong)** By setting $f(y)$ to a constant value
V: **(wrong)** I do not know

**Solution 1:**

Equilibrium points are the values of $y$ where $\dot{y} = f(y) = 0$. These are points where the system remains at rest if not perturbed.

- building and interpreting phase portraits 3

notes

- see the associated solution(s), if compiled with that ones :)

## Question 3

Which of the following best describes the phase portrait of the system $\dot{y} = y(1 - y)$?

**Potential answers:**

I: **(wrong)** It consists of a single trajectory with no equilibrium points
II: **(correct)** It has two equilibrium points at $y = 0$ and $y = 1$, with flow directions determined by the sign of $f(y)$
III: **(wrong)** It has infinitely many equilibrium points
IV: **(wrong)** It has no equilibrium points and exhibits oscillatory behavior
V: **(wrong)** I do not know

**Solution 1:**

The function $f(y) = y(1 - y)$ has two roots at $y = 0$ and $y = 1$, which are the equilibrium points. The direction of flow depends on the sign of $f(y)$.



notes

- see the associated solution(s), if compiled with that ones :)

## Question 4

What distinguishes the phase portrait of a second-order system from a first-order system?

**Potential answers:**

I: **(wrong)** Second-order phase portraits only have one equilibrium point
II: **(correct)** Second-order phase portraits require a two-dimensional state space (e.g., $x$ vs. $\dot{x}$)
III: **(wrong)** First-order systems can have limit cycles, while second-order systems cannot
IV: **(wrong)** Phase portraits for second-order systems do not contain information about stability
V: **(wrong)** I do not know

**Solution 1:**



Second-order ODEs require a two-dimensional phase space, where each point represents both a position variable and its derivative. This allows visualization of

notes

- see the associated solution(s), if compiled with that ones :)

## Question 5

Which of the following statements about phase portraits of nonlinear systems is correct?

**Potential answers:**

I: **(wrong)** Nonlinear systems always have a single equilibrium point
II: **(wrong)** Nonlinear phase portraits can be analyzed only by solving the system numerically
III: **(correct)** Nonlinear phase portraits may exhibit equilibrium points, limit cycles, and chaotic behavior
IV: **(wrong)** Nonlinear phase portraits always resemble those of linear systems for small perturbations
V: **(wrong)** I do not know

**Solution 1:**

Nonlinear systems can have equilibrium points, periodic limit cycles, or even chaotic behavior depending on the dynamics. Their phase portraits often exhibit richer and more complex structures than linear systems.



notes

- see the associated solution(s), if compiled with that ones :)

## Recap of sub-module "building and interpreting phase portraits"

- A phase portrait is a graphical representation of a dynamical systems trajectories in state space.
- Phase portraits provide qualitative insight into system behavior without requiring explicit solutions.
- First-order systems have a one-dimensional state space, while second-order systems require two dimensions, etc.



notes

- the most important remarks from this sub-module are these ones