Systems Laboratory, Spring 2025

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notes

- welcome to the course!
- on this side of this document you will find notes that accompany the text typically visualized in class
- these notes are meant to convey the messages that are not displayed in the text on the side, and basically constitute what the teacher intends to say in class

Table of Contents I

notes

- this is the table of contents of this document; each section corresponds to a specific part of the course compute the equilibria of the system

notes

Contents map

developed content units	taxonomy levels
equilibrium	u1, e1

prerequisite content units	taxonomy levels
ODE	u1, e1

notes

Main ILO of sub-module "compute the equilibria of the system"

Compute the equilibria of an ODE by solving for stationary points

notes

- by the end of this module you shall be able to do this

Is this in equilibrium?

notes

- yes, intuitively

Are these in equilibrium, while falling?

notes

- no, intuitively

Equilibrium = a trajectory that is constant in time

$$\dot{y}(t) = 0$$

notes

- the simplest trajectories are that ones that seem simple constant numbers

Example

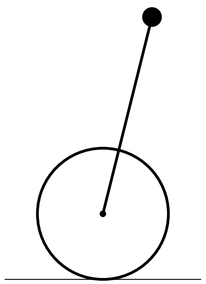
temperature of a small brick in a very large room whose temperature is 20 degrees:

$$\dot{T} = -0.5\left(T - 20\right)$$

notes

- let's do a practical example of a simple system
- here we can see that starting from any T we ideally tend to go to T_a
- and if we start there we have the derivative equal to zero, thus an equilibrium

What does it mean that this system is in equilibrium from an intuitive point of view?



notes

- let's start with the simplest concept, the one of equilibrium, through a practical example: a segway is in equilibrium if it is perfectly upright or if it is laying on the floor
- the term equilibrium means, from an intuitive point of view, 'things do not move', and thus the trajectories stay constant

What does it mean that this system is in equilibrium from a mathematical point of view?

$$\boxed{\text{equilibrium means } \dot{\boldsymbol{y}} = \mathbf{0}}$$

this implies

$$\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}} \text{ is an equilibrium point iff } \dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}\right) = \mathbf{0}$$

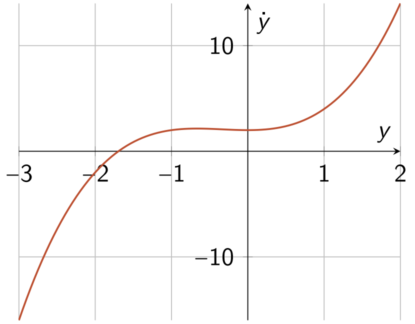
i.e., the equilibria of a system are the zeros of $\boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right)$

notes

- 'things do not move' then mathematically translate into $\dot{\boldsymbol{y}} = \mathbf{0}$
- $\dot{\boldsymbol{y}} = \mathbf{0}$ means that we are looking for that points that make $\boldsymbol{f}$ zero
- note that "iff" means if and only if

Equilibra as the zeros of $\boldsymbol{f}$, graphically

Exemplified situation of *autonomous* single output systems:

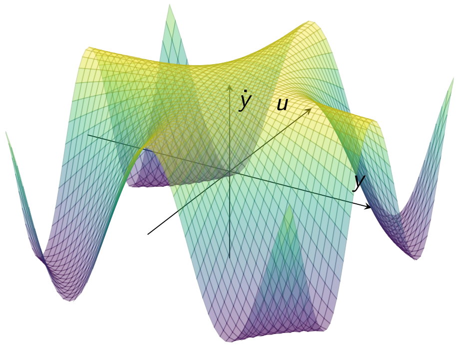


notes

- so one can see the equilibria from the crossings of $\boldsymbol{f}$ with the 'x axis' (note that this is a scalar case, and the "autonomous" case where there is no input u, so a simplified case)

Equilibra as the zeros of $\boldsymbol{f}$, graphically

Exemplified situation of SISO (single input single output) systems:



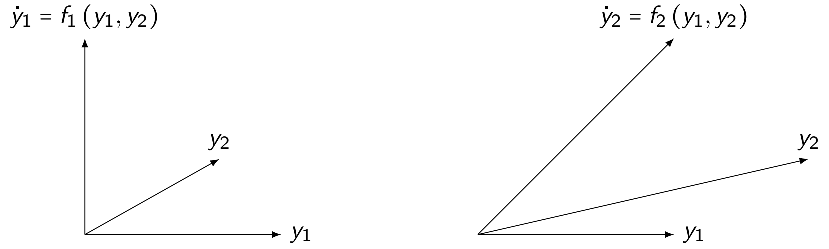
https://www.geogebra.org/classic/mmppe6hs

notes

- one can see the equilibria from the crossings of $\boldsymbol{f}$ with the 'yu plane' in case we have more dimensions

Equilibra as the zeros of $\boldsymbol{f}$, graphically

Exemplified situation of automonous multiple output systems:



(remember: $\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}$ equilibrium iff $\boldsymbol{f}(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) = \mathbf{0}$, i.e., all the components simultaneously!)

notes

- note that if one has more than one y then it must be $\dot{y}_i = 0$ for all the i's simultaneously at the same point equilibria = extremely important topic in automatic control!

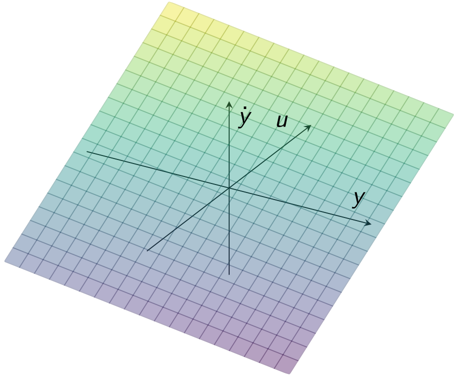
will be analysed in more details later on in this course
and much more extensively in others
(feat. Lyapunov, Krasovskii, La-Salle among others)

notes

- this module is on ODEs and not on equilibria; however better clarifying that this is an extremely important thing

Discussion: what are the equilibria in this case?

$$\dot{y} = ay + bu \tag{1}$$



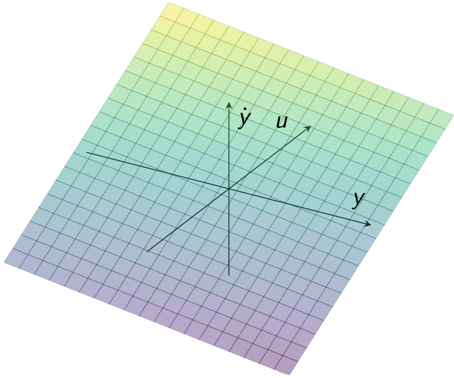
$(y_0 = 0, u = 0)$ is always an equilibrium for these systems!

notes

-
- $\dot{y} = 0$, so if $a = 0$ then any y but it has to be $u = 0$, if $a \neq 0$ then it has to be so that $ay + bu = 0$
- it is also interesting to note this

Discussion: can we have for this specific ODE an equilibrium if $u \neq \mathrm{constant}$?

$$\dot{y} = ay + bu \tag{2}$$



notes

- no

Another example: a Lotka-Volterra model ($\neq$ real world):

$$\begin{cases} \dot{y}_{\text{rabbits}} & = \ \ 0.4 \cdot y_{\text{rabbits}} - 0.5 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \\ \dot{y}_{\text{foxes}} & = -3 \cdot y_{\text{foxes}} + 0.7 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \end{cases}$$

notes

- this you should be immediately able to find an equilibrium, that is 0,0. The other one there is a bit of algebra to do but it comes natually

Summarizing

Compute the equilibria of an ODE by solving for stationary points

- put $f = 0$ and compute the corresponding points. It may be that there is the need to put $u = \mathrm{constant}$

notes

- by the end of this module you shall be able to do this

Most important python code for this sub-module

notes

Root finding in python

https://pythonnumericalmethods.studentorg.berkeley.edu/notebooks/chapter19.05-Root-Finding-in-Python.html

notes

- this page explains how to find the zeros

Self-assessment material

notes

Question 1

What is the mathematical definition of an equilibrium point for a dynamical system $\dot{y} = f(y, u)$?

Potential answers:

I: **(wrong)** A point where $f(y, u)$ is maximized
II: **(correct)** A point where $f(y, u) = 0$
III: **(wrong)** A point where y is always increasing
IV: **(wrong)** A point where y is always decreasing
V: **(wrong)** I do not know

Solution 1:

An equilibrium point is defined as a point where the system's derivative $\dot{y}$ is zero, meaning the state remains constant over time.

notes

- see the associated solution(s), if compiled with that ones :)

Question 2

Which of the following statements about equilibrium points is correct?

Potential answers:

I: **(wrong)** An equilibrium point is always stable
II: **(correct)** An equilibrium point is where the system's state does not change over time
III: **(wrong)** An equilibrium point is a location where external inputs are irrelevant
IV: **(wrong)** An equilibrium point always corresponds to $y = 0$
V: **(wrong)** I do not know

Solution 1:

An equilibrium point is defined as a state where the time derivative of the system is zero, meaning the system does not evolve from that state unless disturbed. Stability depends on additional conditions.

notes

- see the associated solution(s), if compiled with that ones :)

Question 3

Graphically, how can equilibrium points be identified for an autonomous system $\dot{y} = f(y)$?

Potential answers:

I: **(wrong)** By finding where $y = 0$
II: **(correct)** By finding the points where $f(y) = 0$ on the phase plot
III: **(wrong)** By locating the steepest points of the function $f(y)$
IV: **(wrong)** By identifying the points where y reaches its maximum or minimum values
V: **(wrong)** I do not know

Solution 1:

Equilibrium points occur where $\dot{y} = f(y) = 0$, which correspond to the intersections of $f(y)$ with the horizontal axis in a phase plot.

notes

- see the associated solution(s), if compiled with that ones :)

Question 4

Consider the system $\dot{T} = -0.5(T - 20)$. What is the equilibrium temperature?

Potential answers:

I: **(wrong)** $T = 0$
II: **(correct)** $T = 20$
III: **(wrong)** $T = -20$
IV: **(wrong)** $T = 40$
V: **(wrong)** I do not know

Solution 1:

Setting $\dot{T} = 0$ gives $-0.5(T - 20) = 0$, which simplifies to $T = 20$. This is the equilibrium point of the system.

notes

- see the associated solution(s), if compiled with that ones :)

Question 5

For the linear system $\dot{y} = ay + bu$, under what condition is (y_0, u_0) an equilibrium?

Potential answers:

I: **(wrong)** When $a = 0$ only
II: **(correct)** When $ay_0 + bu_0 = 0$
III: **(wrong)** When $y_0 = 0$ and $u_0 = 0$ always
IV: **(wrong)** When u_0 is arbitrary
V: **(wrong)** I do not know

Solution 1:

To find an equilibrium, set $\dot{y} = 0$, leading to $ay + bu = 0$. Solving for y_0 and u_0 gives the equilibrium condition.

notes

- see the associated solution(s), if compiled with that ones :)

Question 6

If we have an autonomous time-varying ODE, can we have equilibria?

Potential answers:

I: **(wrong)** No, time-variation always prevents equilibria.
II: **(correct)** Yes, equilibria can exist if the system allows constant solutions.
III: **(wrong)** Only if the system is also linear.
IV: **(wrong)** Yes, but only if the system is also periodic.
V: **(wrong)** I do not know

Solution 1:

Equilibria exist when the derivative of the state variables is zero. Even though the system is time-varying, there can still be points where the vector field is zero, leading to equilibrium. For example, we may have $\dot{y} = ty$. When $y = 0$ we have an equilibrium anyway.

notes

- see the associated solution(s), if compiled with that ones :)

Question 7

Can we have dynamical systems that do not have any equilibria?

Potential answers:

I: **(wrong)** No, every system has at least one equilibrium.
II: **(correct)** Yes, with no fixed points may lack equilibria.
III: **(wrong)** Only non-autonomous systems can lack equilibria.
IV: **(wrong)** No, because every system must have at least a trivial equilibrium.
V: **(wrong)** I do not know

Solution 1:

Very simple example: $\dot{y} = 1$.

notes

- see the associated solution(s), if compiled with that ones :)

Question 8

If we have a non-autonomous ODE, can we have equilibria if the input is always changing, e.g., $u = \sin(t)$?

Potential answers:

I: **(wrong)** Yes, the input does not affect equilibrium conditions.
II: **(wrong)** No, because a changing input continuously affects system states.
III: **(wrong)** Only if the input has a zero mean.
IV: **(wrong)** Yes, but only if the system is linear.
V: **(correct)** Yes, and the system does not need to be linear.
VI: **(wrong)** I do not know

Solution 1:

A simple example: $\dot{y} = yu$. For $y = 0$ we have for sure an equilibrium independently of u.

notes

- see the associated solution(s), if compiled with that ones :)

Question 9

If we have a non-autonomous LTI ODE, can we have equilibria if the input is always changing, e.g., $u = \sin(t)$?

Potential answers:

- I: **(wrong)** Yes, because LTI systems always have equilibria.
- II: **(correct)** No, because the continuously varying input prevents a steady state.
- III: **(wrong)** Only if the system has no damping.
- IV: **(wrong)** Yes, but only if the input is periodic.
- V: **(wrong)** I do not know

Solution 1:

For an LTI system, equilibrium requires all derivatives to be zero. A continuously varying input like $u = \sin(t)$ forces the system to evolve dynamically, preventing equilibrium.

notes

- see the associated solution(s), if compiled with that ones :)

Recap of sub-module "compute the equilibria of the system"

- Equilibria in dynamical systems correspond to points where the system's state does not change over time.
- Autonomous time-varying ODEs can have equilibria, but their location may vary with time.
- Some dynamical systems may not have equilibria, particularly if they involve unbounded growth.
- Non-autonomous LTI ODEs can have equilibria only if the input $u(t)$ remains constant over time.

notes

- the most important remarks from this sub-module are these ones