Introduction to Luenberger observers

Contents map

developed content units	**taxonomy levels**
observer	u1, e1

prerequisite content units	**taxonomy levels**
feedback	u1, e1
continuous time LTI systems	u1, e1

Main ILO of sub-module "Introduction to Luenberger observers"

Derive the error dynamics equation for state estimators and explain its significance for observer stability

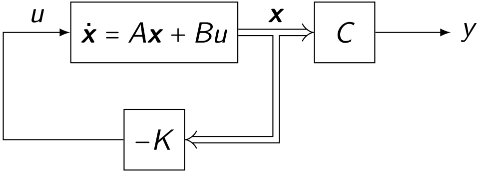
Describe the meaning of the gain matrix L in Luenberger observers

List rules of thumb to select estimator poles based on controller dynamics and system characteristics

Discuss the trade-offs between fast and slow observers in the presence of process and measurement noise

Control-law design for full-state feedback

from



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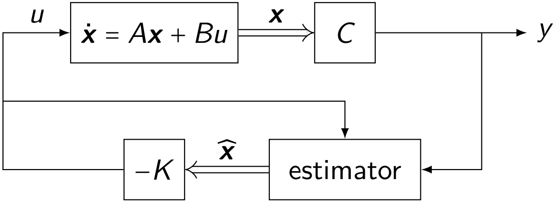
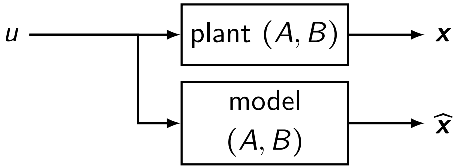


First idea for how to estimate $\boldsymbol{x}$: open loop estimator

$$\text{I know} \begin{cases} \boldsymbol{x}(0) \\ u(k) \\ A, B \end{cases}$$

$$\implies \quad \text{I can simulate} \begin{cases} \widehat{\boldsymbol{x}}(0) = \boldsymbol{x}(0) \\ \dot{\widehat{\boldsymbol{x}}} = A\widehat{\boldsymbol{x}} + Bu \end{cases}$$



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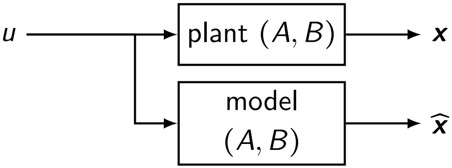

Dynamics:

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \qquad \widehat{\boldsymbol{x}}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

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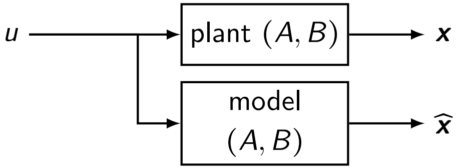

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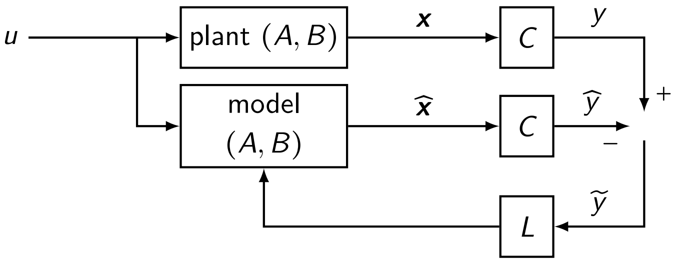
though, is this strategy robust? *no! if there is uncertainty the estimation error* $\widetilde{\boldsymbol{x}}(t) := \boldsymbol{x}(t) - \widehat{\boldsymbol{x}}(t)$ *may diverge*

Estimator design

Idea: use feedback



$$\begin{cases} \widehat{\boldsymbol{x}}(0) = \boldsymbol{x}(0) \\ \dot{\widehat{\boldsymbol{x}}} = A\widehat{\boldsymbol{x}} + Bu + L\left(y - C\widehat{\boldsymbol{x}}\right) \end{cases}$$

What are the dynamics of the error $\widetilde{\boldsymbol{x}} := \boldsymbol{x} - \widehat{\boldsymbol{x}}$?

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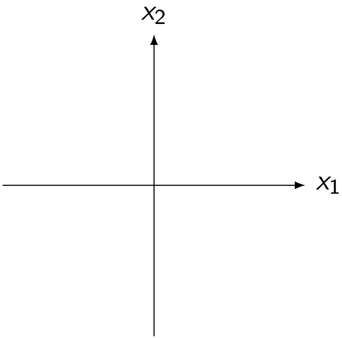
$$\Downarrow$$

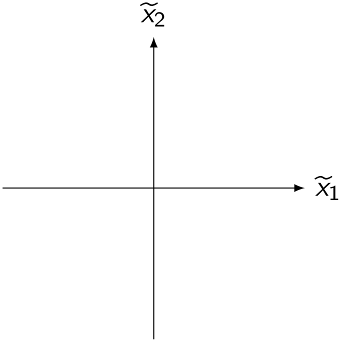
$$\dot{\widetilde{\boldsymbol{x}}} = \left(A - LC\right)\widetilde{\boldsymbol{x}}$$

What decides the stability and speed of the dynamics of the error? *The eigenvalues of* $A - LC$*!*

Dynamics of the error vs. dynamics of the state

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \qquad \dot{\widetilde{\boldsymbol{x}}} = (A - LC)\,\widetilde{\boldsymbol{x}}$$





How can we design L?

For $(A, B, C, 0)$ fully observable & in observation canonical form:

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

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$$\det(sI - A) = s^n + a_1 s^{n-1} + \dots + a_n$$

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$$L = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{bmatrix} \quad \Longrightarrow \quad LC = \begin{bmatrix} L_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ L_n & 0 & \cdots & 0 \end{bmatrix}$$

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$$\det(sI - (A - LC)) = s^n + (a_1 + L_1)s^{n-1} + \dots + (a_n + L_n)$$

Using acker for designing L

```
% controller design
K = acker( A, B, afPoles );

% observer design
L = ( acker( A', C', afPoles ) )';
```

how to select the poles of the estimator

Estimator poles selection – rules of thumb

- the observer should be $2 \rightarrow 6$ times faster than the controller

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Estimator poles selection – rules of thumb

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- the more the sensors are noisy, the slower the observer should be
- the slower the observer, the less resilient the controller is to disturbances

algorithms for designing L:

- dominant second order poles
- LQR

Estimator poles selection – connections with estimation theory

$$\begin{cases} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + Bu \\ y &= C\boldsymbol{x} \end{cases} \quad \mapsto \quad \begin{cases} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + Bu + \boldsymbol{w} \\ y &= C\boldsymbol{x} + \nu \end{cases}$$

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Estimator as before:

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New error dynamics:

$$\dot{\widetilde{\boldsymbol{x}}} = \left(A - LC\right)\widetilde{\boldsymbol{x}} + \boldsymbol{w} - L\nu$$

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Trade-off:

- L big $\implies$ effect of $\boldsymbol{w}$ is negligible but ν is amplified
- L small $\implies$ effect of ν is negligible but $\boldsymbol{w}$ has bigger influences

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optimal strategy in a statistical sense requires Kalman filtering (not in this course)

Summarizing

Derive the error dynamics equation for state estimators and explain its significance for observer stability

Describe the meaning of the gain matrix L in Luenberger observers

List rules of thumb to select estimator poles based on controller dynamics and system characteristics

Discuss the trade-offs between fast and slow observers in the presence of process and measurement noise

Self-assessment material

Question 1

Why is an open-loop estimator generally not considered a robust method for state estimation?

Potential answers:

I: Because it can track the states accurately even with uncertainties.

II: Because it uses feedback to correct errors in real time.

III: Because in the presence of model uncertainties or disturbances, the estimation error may diverge.

IV: Because it relies on noisy measurements, which destabilize the estimation.

V: I do not know

Question 2

What determines the stability and speed of convergence of the estimation error $\widetilde{\boldsymbol{x}} = \boldsymbol{x} - \widehat{\boldsymbol{x}}$ in a full-state observer?

Potential answers:

I: The eigenvalues of matrix B.
II: The input $u(t)$ and measurement noise.
III: The initial state $\boldsymbol{x}(0)$.
IV: The eigenvalues of the matrix $A - LC$.
V: I do not know

Question 3

In the observer canonical form, how does the gain matrix L affect the characteristic polynomial of the observer error dynamics?

Potential answers:

I: It multiplies the coefficients of the system matrix A.
II: It adds to the coefficients of the characteristic polynomial of A.
III: It replaces the eigenvalues of A with the eigenvalues of B.
IV: It subtracts from the output matrix C to reduce measurement noise.
V: I do not know

Question 4

Why is it generally recommended for the observer poles to be 2 to 6 times faster than the controller poles?

Potential answers:

I: To ensure the controller has enough time to react to the observer.
II: To make the estimation error dynamics faster than the control system, enabling accurate feedback.
III: To slow down the observer and reduce noise amplification.
IV: To match the sampling rate of the digital controller.
V: I do not know

Question 5

What is a trade-off involved when choosing faster poles for the observer?

Potential answers:

I: Faster poles increase sensitivity to measurement noise.
II: Faster poles always improve estimation accuracy, regardless of noise.
III: Slower poles make the observer more responsive.
IV: Faster poles reduce computational complexity.
V: I do not know

Recap of sub-module "Introduction to Luenberger observers"

- one may estimate the states of a system by means of making the estimated state be so that it dynamically matches the measured values
- this strategy though is as valid as the model is, as a description of the system
- the situation is though in practice not as simple as seen here - indeed the here presented case is for "fully observable" systems (a concept that you'll see in systems theory) and thus not applicable all the times (but extensible to!)

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