Full state feedback control

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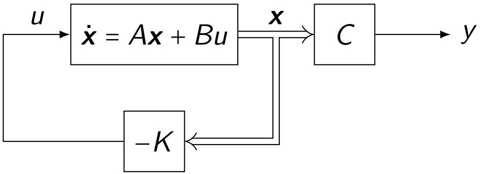
Main ILO of sub-module "Full state feedback control"

Formulate a state feedback control law $u = -K\boldsymbol{x}$ to modify the closed-loop dynamics of a linear time-invariant system, given matrices A and B in state-space form

Compute the matrix K to place the poles of the closed-loop system at specified locations, using characteristic polynomial matching

Apply the pole placement algorithm to determine the feedback matrix K for a system with A, B in control canonical form, using time-domain specifications *note: the considerations below are the same for both discrete time and continuous time LTIs*

Control-law design for full-state feedback – assumed structure



$$u = -K\boldsymbol{x} = -\begin{bmatrix} K_1 & \dots & K_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

(estimating $\boldsymbol{x}$ *from the measurements = later on)*

Finding the control law

$$\begin{cases} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + Bu \\ y &= C\boldsymbol{x} \end{cases} \qquad \text{``}+\text{''} \qquad u = -K\boldsymbol{x}$$

$$\Downarrow$$

$$\begin{aligned} \dot{\boldsymbol{x}} &= (A - BK)\,\boldsymbol{x} \\ y &= C\boldsymbol{x} \end{aligned}$$

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Important:

$$BK = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \begin{bmatrix} K_1 & \dots & K_n \end{bmatrix} = \begin{bmatrix} b_1 K_1 & \cdots & b_1 K_n \\ \vdots & & \vdots \\ b_n K_1 & \cdots & b_n K_n \end{bmatrix}$$

Finding the control law – what are the poles now?

$$\begin{aligned} \dot{\boldsymbol{x}} &= (A - BK)\,\boldsymbol{x} \\ y &= C\boldsymbol{x} \end{aligned} \quad \Rightarrow \quad \det\Big(sI - (A - BK)\Big) = 0$$

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choose K *so that the closed-loop poles are where we like*

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Poles allocation algorithm

- from time-domain specifications, numerically determine the n desired poles $p_1, \ldots, p_n$

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- find $K_1, \ldots, K_n$ by equating the two polynomials

Example

Close the loop around the open loop system $\left\{ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \right.$ so that the closed loop is stable and with raise time not longer than 5 seconds

$$\left(\text{recall:} \quad G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \rightarrow \quad \text{rise time } t_r = \frac{1.8}{\omega_n} \right)$$

Finding the control law – drawback when A has no special structure

Example:

$$A = \begin{bmatrix} 5 & 1 & 3 & 8 & 3 \\ 7 & 3 & 9 & 6 & 9 \\ 9 & 4 & 4 & 1 & 7 \\ 2 & 2 & 5 & 3 & 6 \\ 1 & 1 & 0 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \\ 9 \end{bmatrix}$$

drawback: doing as before is cumbersome

$\downarrow$

is there any alternative way of finding K?

Determinant of a matrix in control canonical form

Let

$$A = \begin{bmatrix} -a_1 & -a_2 & \dots & \dots & -a_n \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \end{bmatrix}$$

then

$$\det\left(sI - A\right) = s^n + a_1 s^{n-1} + \dots + a_n$$

Incidentally...

$$Y(s) = \frac{b_1 s^{n-1} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n} U(s)$$

$$\mapsto \quad A = \begin{bmatrix} -a_1 & -a_2 & \ldots & \ldots & -a_n \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \end{bmatrix}$$

$$\mapsto \quad \det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_n$$

Finding the control law with (A, B) in control canonical form

$$B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad BK = \begin{bmatrix} K_1 & K_2 & \cdots & K_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

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so that

$$A - BK = \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & \dots & \dots & -a_n - K_n \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \end{bmatrix}$$

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and thus the poles of the closed loop system are the roots of

$$\det\left(sI - (A - BK)\right) = s^n + (a_1 + K_1)\, s^{n-1} + \ldots + (a_n + K_n)$$

Summary (valid also for discrete-time systems!)

$$(A, B) \text{ in control canonical form } + K \text{ generic}$$

$$\Downarrow$$

$$\text{closed loop is } \dot{\boldsymbol{x}} = (A - BK)\boldsymbol{x} = \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & \dots & \dots & -a_n - K_n \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x}$$

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poles of the closed loop system = roots of

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- from time domain specifications, find the desired poles $p_1, \ldots, p_n$

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- from time domain specifications, find the desired poles $p_1, \ldots, p_n$
- form the desired characteristic polynomial

$$\alpha(s) = \prod_{i=1}^{n} (s - p_i) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$$

Summary of the algorithm for (A, B) in control canonical form

- from time domain specifications, find the desired poles $p_1, \ldots, p_n$
- form the desired characteristic polynomial

$$\alpha(s) = \prod_{i=1}^{n} (s - p_i) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$$

- find K s.t. $\det(sI - A + BK) = \alpha(s)$ by solving

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} a_1 + K_1 \\ \vdots \\ a_n + K_n \end{bmatrix}$$

Example

Close the loop around the discrete time open loop system $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
with sampling period 0.2 seconds so that the closed loop raise time is not longer than 10 seconds

$$\left(\text{recall:} \qquad G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \to \quad \text{rise time } t_r = \frac{1.8}{\omega_n} \quad \text{but we need to discretize!} \right)$$

Test this out: write a K that makes the discrete time open loop system

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with sampling period 0.5 seconds have a raise time is not longer than 15 seconds.

Fundamental difference with PIDs

$$\det\left(sI - (A - BK)\right) = s^n + (a_1 + K_1)\, s^{n-1} + \ldots + (a_n + K_n)$$

state-feedback in fully controllable systems allows allocating all the closed loop poles wherever one wants

Caveats

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- weak controllability
- the more you move poles, the more you use actuators (risk of saturations!)
- effect of zeros

Do we actually need to compute the control canonical form?

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no, there exists the so-called Ackermann's formula

$$K = [0 \ \dots \ 0 \ 1] \, \mathcal{C}^{-1} \alpha(A)$$

we will though do not cover it - will be in follow up courses!

But how do we select the locations of the poles?

Strategies:

- *in this course,* dominant second-order poles approximations
- *in follow up courses,* very many other ones!

Summarizing

Formulate a state feedback control law $u = -K\boldsymbol{x}$ to modify the closed-loop dynamics of a linear time-invariant system, given matrices A and B in state-space form

Compute the matrix K to place the poles of the closed-loop system at specified locations, using characteristic polynomial matching

Apply the pole placement algorithm to determine the feedback matrix K for a system with A, B in control canonical form, using time-domain specifications

Most important python code for this sub-module

control (Python Control Systems Library)

main functions:

- `acker` (Ackermann's method)
- `place` (robust pole placement)

Self-assessment material

Question 1

What is the primary advantage of state feedback control with pole placement compared to PID control?

Potential answers:

I: PID control is always more stable than state feedback.
II: State feedback allows arbitrary placement of all closed-loop poles when the system is fully controllable.
III: State feedback does not require knowledge of the system's state variables.
IV: PID control can achieve faster response times than state feedback.
V: I do not know

Question 2

Why is the control canonical form particularly useful for pole placement problems?

Potential answers:

I: It makes the system matrix A diagonal.
II: It eliminates all zeros from the transfer function.
III: The coefficients of the characteristic polynomial appear directly in the first row of A.
IV: It guarantees that the system will be observable.
V: I do not know

Question 3

What is a major practical limitation of aggressive pole placement through state feedback?

Potential answers:

I: It makes the system uncontrollable.
II: It may require large control inputs that could lead to actuator saturation.
III: It always makes the system unstable.
IV: It prevents the use of output feedback.
V: I do not know

Question 4

When designing state feedback control, why might we choose poles with dominant second-order characteristics?

Potential answers:

I: Because higher-order systems cannot be controlled effectively.

II: Because it eliminates all zeros from the system.

III: Because it allows us to approximate the response using familiar second-order performance measures.

IV: Because it guarantees minimum-phase behavior.

V: I do not know

Recap of sub-module “Full state feedback control”

- full state feedback enables placing the poles wherever one wants
- with respect to PID it has more flexibility
- this comes with the cost of having a sufficiently accurate model (and that the model can be written in control canonical form, something that is not always guaranteed!)

?