Ill conditioning

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ill conditioning	u1, e1
ill posedness	u1, e1

prerequisite content units	taxonomy levels
least squares regression	u1, e1

Main ILO of sub-module "Ill conditioning"

Describe what ill conditioning and ill posedness mean, in the context of system identification

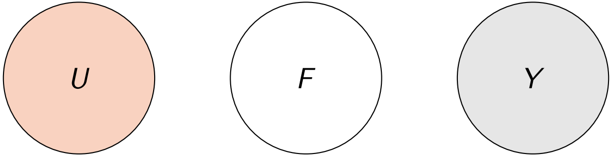
Recognize when ill conditioning may happen in practice

Starting point: system identification

starting from

$$y[k] = f\Big(u[k], u[k-1], \dots\Big) + d[k] \qquad \mathcal{D} = \{u[k], y[k]\}_{k \in \mathcal{K}}$$

identify the model $f(\cdot)$



Definition of ill-posedness and ill-conditioning

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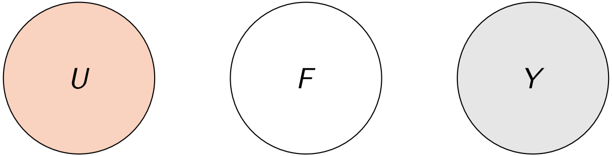

- *ill-posed problem (in the Hadamard sense):* solution is either not unique or does not depend continuously on the data

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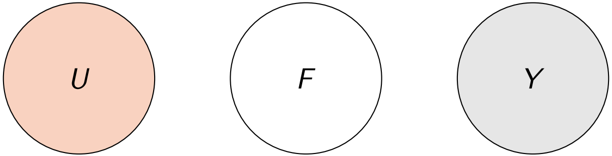
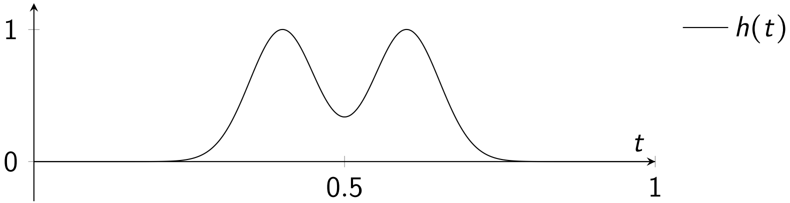


- *ill-posed problem (in the Hadamard sense):* solution is either not unique or does not depend continuously on the data
- *ill-conditioned problem:* solution is very sensitive to the data

Example: the Hunt reconstruction problem

(continuous-time LTI with sampled output)

$$h(t) = \exp\left(-\left(\frac{t-0.4}{0.075}\right)^2\right) + \exp\left(-\left(\frac{t-0.6}{0.075}\right)^2\right)$$



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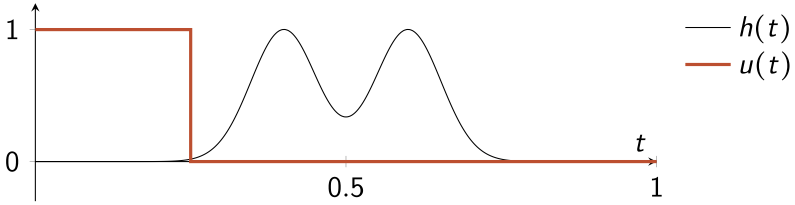

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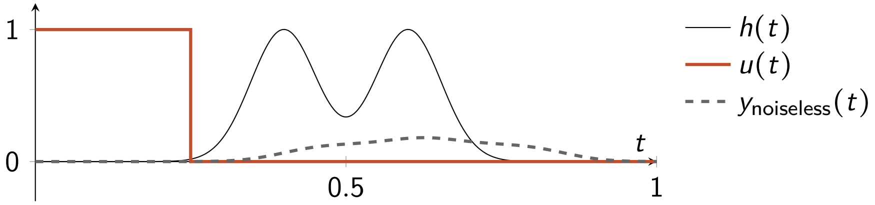
$$y_{\text{noiseless}}(t) = \int_0^{+\infty} h(\tau)u(t-\tau)d\tau$$



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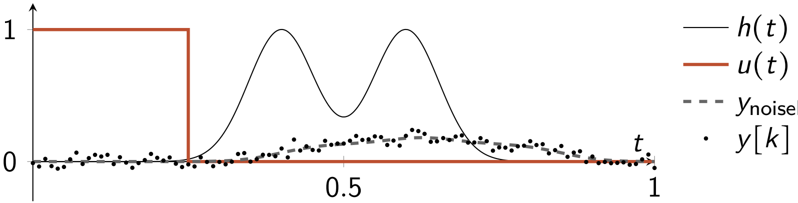
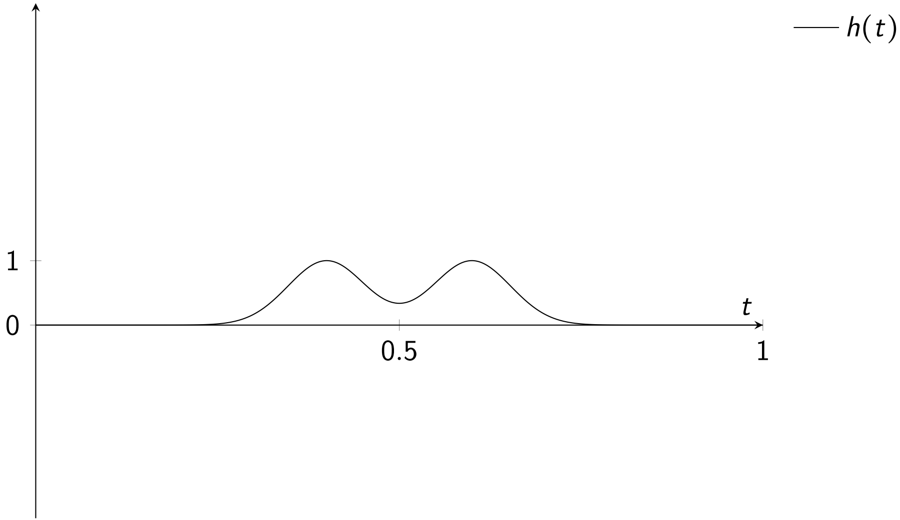
$$y_{\text{noiseless}}(t) = \int_0^{+\infty} h(\tau)u(t-\tau)d\tau \qquad y[k] = y_{\text{noiseless}}[k] + v(k)$$



This problem can be solved with linear algebra!

$$\boldsymbol{y} = U\boldsymbol{h} + \boldsymbol{d} \quad \Rightarrow \quad \widehat{\boldsymbol{h}} = \left(U^T U\right)^{-1} U^T \boldsymbol{y}$$

Is the Hunt reconstruction problem well defined?



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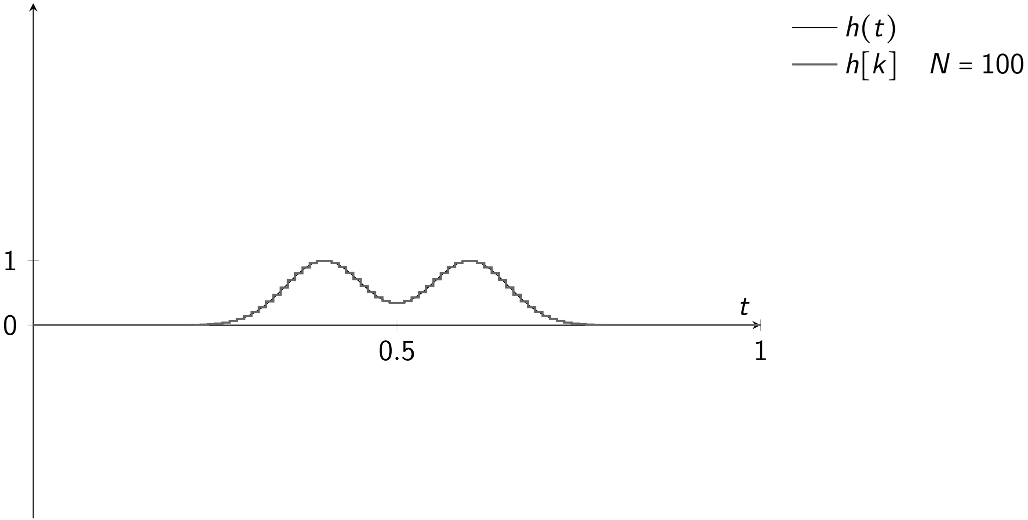

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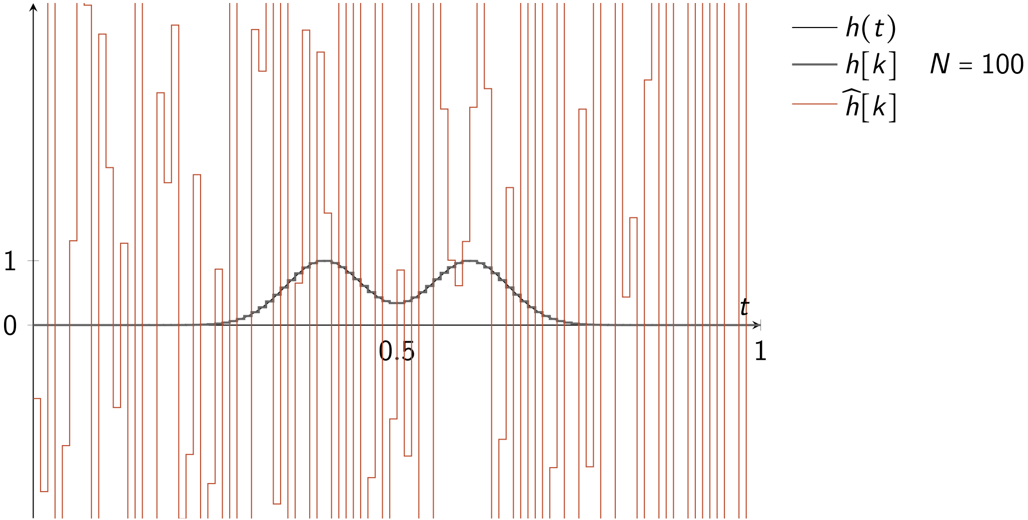

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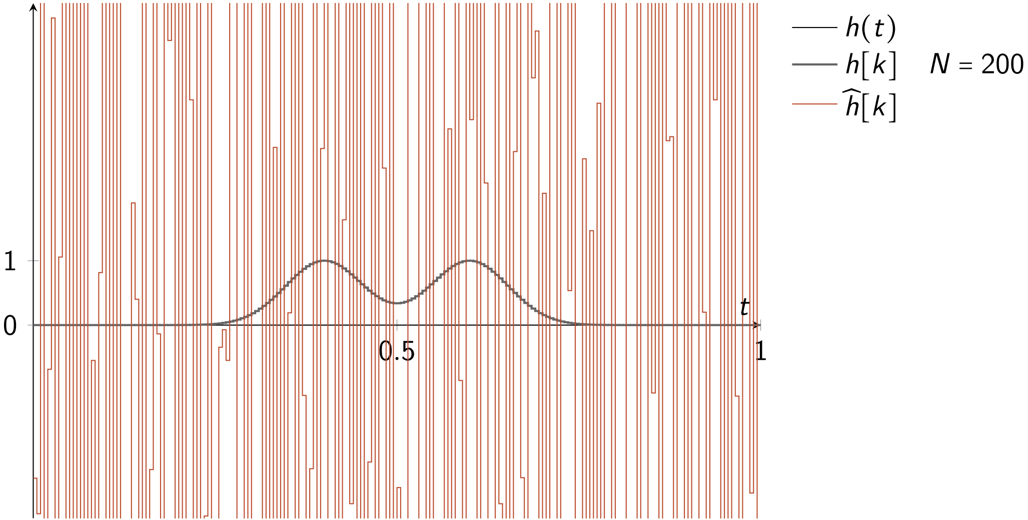

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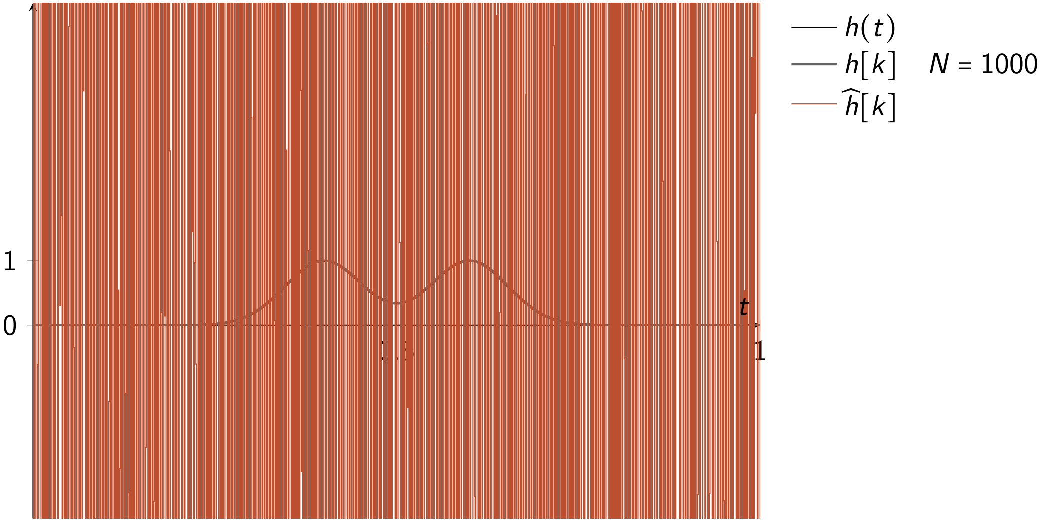

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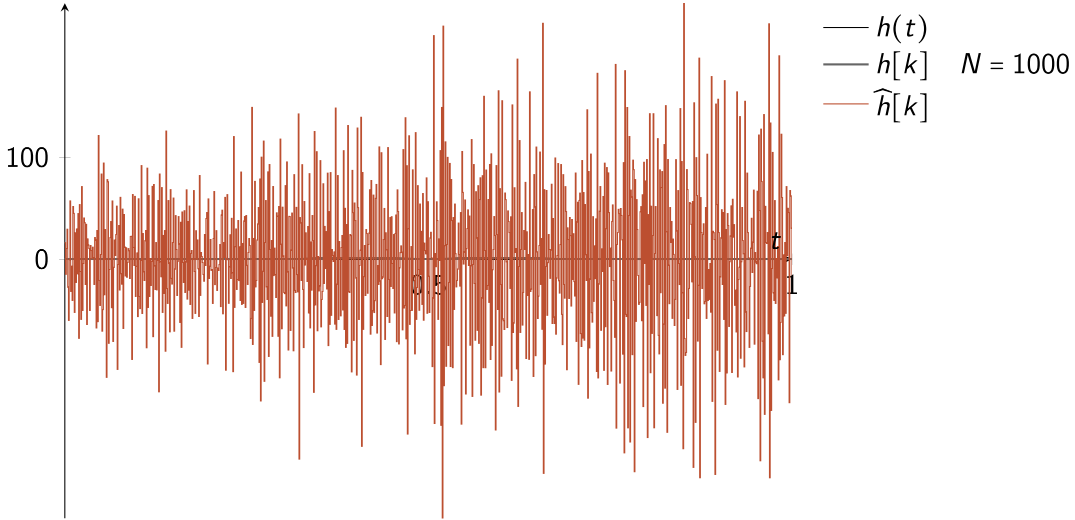

What is happening?

$$\boldsymbol{e} = \boldsymbol{h} - \widehat{\boldsymbol{h}} = U^{-1}\boldsymbol{d}$$

$$\frac{\|\boldsymbol{e}\|}{\|\boldsymbol{h}\|} \leq \frac{\sigma_{\max}(U)}{\sigma_{\min}(U)} \frac{\|\boldsymbol{d}\|}{\|U\boldsymbol{h}\|}$$

- the slower u the higher $\frac{\sigma_{\max}(U)}{\sigma_{\min}(U)}$
- the faster Δ the higher $\frac{\sigma_{\max}(U)}{\sigma_{\min}(U)}$

how can we improve our estimates?

$\mapsto$ regularization

Summarizing

Describe what ill conditioning and ill posedness mean, in the context of system identification

Recognize when ill conditioning may happen in practice

- TODO

Most important python code for this sub-module

Linear algebra tools

- `numpy.linalg.solve`
- `numpy.linalg.inv`

Self-assessment material

Question 1

Which of the following best describes the difference between an ill-posed and an ill-conditioned problem in system identification?

Potential answers:

I: Ill-conditioned problems have no solution, while ill-posed problems have too many.

II: Ill-posed problems may lack uniqueness or continuous dependence on the data, while ill-conditioned problems are extremely sensitive to small changes in data.

III: Ill-posed problems always have unstable solutions, while ill-conditioned ones always diverge.

IV: Ill-conditioning is due to randomness in the input, while ill-posedness is due to measurement noise.

V: I do not know

Question 2

Why does the Hunt reconstruction problem become ill-conditioned as the length of the input increases?

Potential answers:

I: Because more data always makes the system overdetermined.

II: Because slow or non-diverse input signals lead to poor numerical conditioning of the matrix U.

III: Because increasing the number of samples reduces the noise-to-signal ratio.

IV: Because the model structure becomes nonlinear with large N.

V: I do not know

Question 3

In the context of system identification, what does the condition number $\frac{\sigma_{\max}(U)}{\sigma_{\min}(U)}$ represent?

Potential answers:

I: The maximum amplification of relative errors in the data to the estimation error.
II: The rate of convergence of the optimization algorithm used.
III: The ratio between input and output power in the system.
IV: The likelihood that a model is nonlinear.
V: I do not know

Question 4

What is a practical way to reduce ill-conditioning in system identification?

Potential answers:

I: Use richer or faster-varying input signals during data collection.
II: Use fewer data points to simplify the estimation problem.
III: Reduce the noise artificially in the measurements after data collection.
IV: Make the input signal constant over time to ensure stability.
V: I do not know

Question 5

Why is regularization used when solving ill-conditioned system identification problems?

Potential answers:

I: To make the inverse of U exactly equal to zero.

II: To stabilize the solution by penalizing large parameter values or enforcing smoothness.

III: To reduce the condition number by artificially shrinking the data.

IV: To avoid computing the inverse of the matrix altogether.

V: I do not know

Recap of sub-module “Ill conditioning”

- Ill-posed problems may lack a solution, have multiple solutions, or be highly sensitive to small changes in data
- Ill-conditioned problems have a solution, but it is numerically unstable and highly sensitive to input errors
- The condition number of a matrix quantifies the degree of ill-conditioning; a high condition number indicates poor numerical stability
- In system identification, slowly varying or insufficiently rich input signals can lead to ill-conditioning
- Regularization techniques can mitigate the effects of ill-conditioning by introducing stability through additional constraints
- Choosing appropriate input signals is critical to ensuring well-posed and well-conditioned identification problems
- Understanding the structure and properties of the data matrix (e.g., U in least squares problems) is essential to diagnose ill-conditioning

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