explain and determine the marginal stability of an equilibrium

Contents map

developed content units	taxonomy levels
marginally stable equilibrium	u1, e1
simply stable equilibrium	u1, e1

prerequisite content units	taxonomy levels
RR	u1, e1
equilibrium	u1, e1

Main ILO of sub-module
“explain and determine the marginal stability of an equilibrium”

Graphically explain the definition of marginal stability for equilibria and provide graphical insights into its meaning

Identify and **give examples** of systems that have equilibria that are marginally stable or not, and relate these to real-world situations

Determine if an equilibrium is marginally stable or not by inspecting a phase portrait and by analyzing the behavior of the system near the equilibria

?

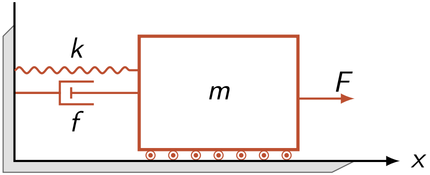
Disclaimer

simply stable = marginally stable

(they are synonyms)

Intuition: if I perturb a little bit this system from its equilibrium, will the system stay closeby or will it go resting to another place?

Example 1



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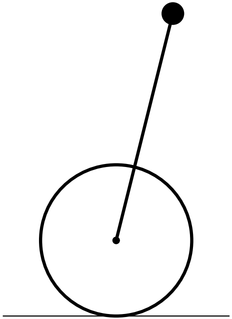
Example 2



Informal introduction to stability

- **asymptotically stable equilibrium:** if I perturb the equilibrium, the system will return there
- **marginally stable equilibrium:** if I perturb the equilibrium, the system will stay around there
- **unstable equilibrium:** if I perturb the equilibrium, the system will move away from it

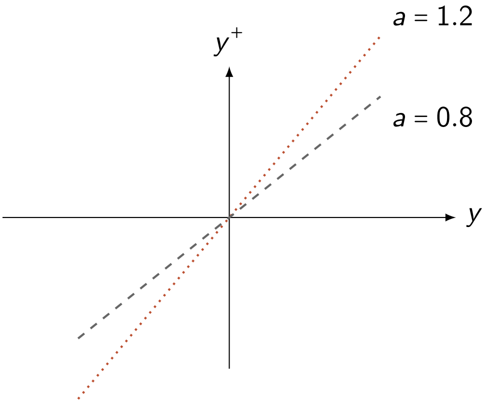
mathematical definitions later on in the program!

stability = extremely important topic in automatic control!

(e.g., will this nuclear plant blow up
if some disturbance slightly perturbs the operating point?)

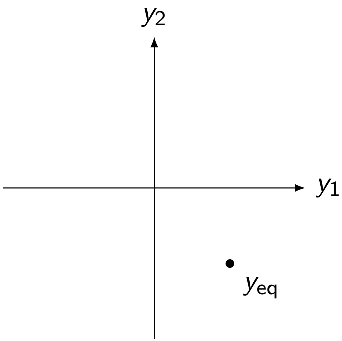
will be analysed in more details later on in this module
and much more extensively in other courses
(feat. Lyapunov, Krasovskii, La-Salle among others)

Understanding the stability properties of $(y_0 = 0, u = 0)$ analysing $y^+ = ay = f(y)$ graphically

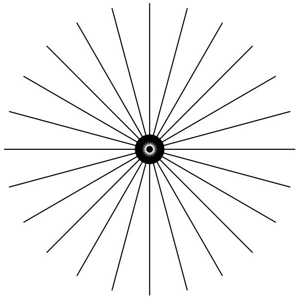


This module = when is an equilibrium marginally stable

overarching question: are we able to bound the trajectories around that equilibrium?



Some phase portraits to exemplify the potential situations



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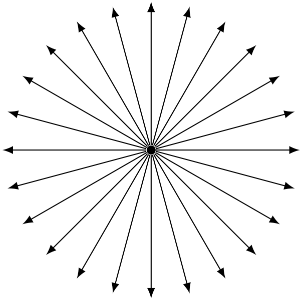

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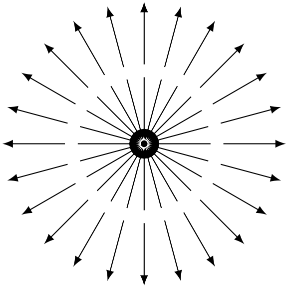

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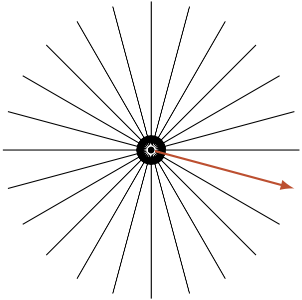

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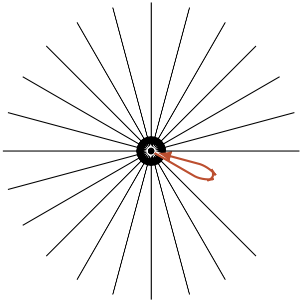
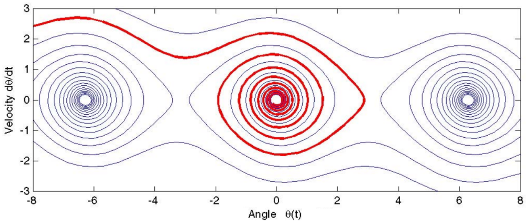


Simply stable equilibrium (discrete time case, formally)



$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$ $\qquad$ $\boldsymbol{y}^+ = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}_e)$ $\qquad$ $(\boldsymbol{y}_e, \boldsymbol{u}_e)$ = equilibrium

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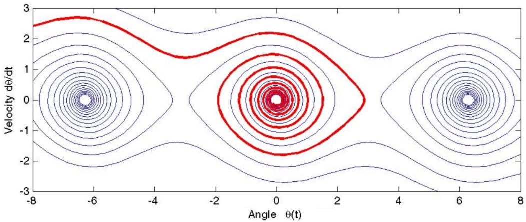

$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$ $\boldsymbol{y}^+ = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}_e)$ $(\boldsymbol{y}_e, \boldsymbol{u}_e)$ = equilibrium

Definition (simply stable equilibrium)

an equilibrium $\boldsymbol{y}_e$ *is simply stable if for any chosen neighborhood of the equilibrium, there exists another one for which trajectories starting in the second neighborhood remain constrained in the first one for all time*

A game to determine if $\boldsymbol{y}_e$ is simply stable or not

Players

you and an “opponent”

A game to determine if $\boldsymbol{y}_e$ is simply stable or not

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Game mechanics

- **step 1** your "opponent" draws a neighborhood containing $\boldsymbol{y}_e$, choosing any neighborhood he/she likes (small, big, whatever, but containing the equilibrium)

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Game mechanics

- **step 1** your “opponent” draws a neighborhood containing $\boldsymbol{y}_e$, choosing any neighborhood he/she likes (small, big, whatever, but containing the equilibrium)
- **step 2** can you find a neighborhood within that neighborhood, again containing $\boldsymbol{y}_e$, so that if the trajectory starts from your neighborhood then it will stay also in the neighborhood of the opponent? *yes* = go back to step 1 and repeat, *no* = you lost and $\boldsymbol{y}_e$ is unstable

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- **exiting the game** can you show that you will win step 2 for any neighborhood that the opponent can choose? Then the equilibrium is marginally stable

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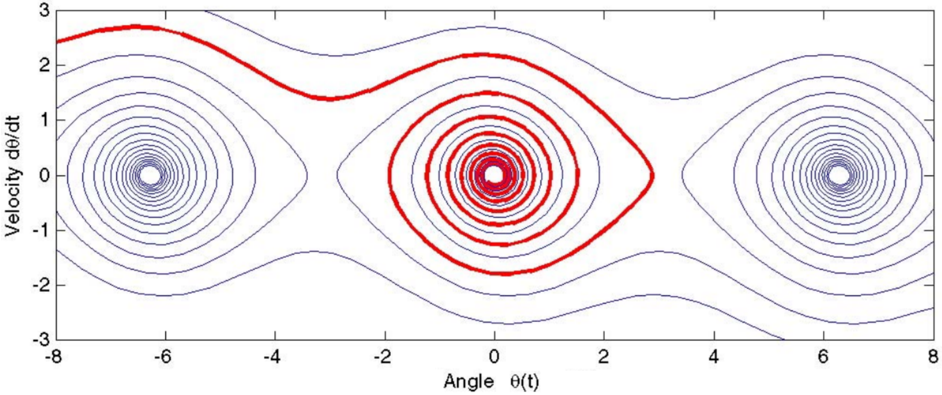
- **exiting the game** can you show that you will win step 2 for any neighborhood that the opponent can choose? Then the equilibrium is marginally stable

important consequence: if an equilibrium is simply stable then it means that I can "confine" the trajectories in any ball of radius ε, with ε a users' choice

Definition of unstable equilibrium

equilibrium = “unstable” if not marginally stable

Example



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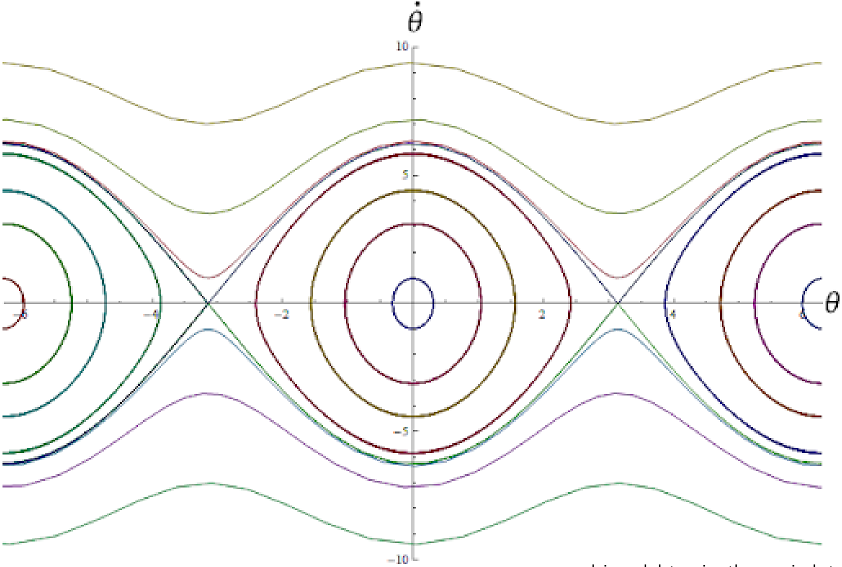

Summarizing

Graphically explain the definition of marginal stability for equilibria and provide graphical insights into its meaning

Identify and **give examples** of systems that have equilibria that are marginally stable or not, and relate these to real-world situations

Determine if an equilibrium is marginally stable or not by inspecting a phase portrait and by analyzing the behavior of the system near the equilibria

- one shall remember the 'game', and the order of who plays first, and who second
- once it is clear that as soon as there is at least one trajectory that "escapes" the equilibrium, that equilibrium is unstable, we got the main point

Most important python code for this sub-module

No dedicated python toolbox for tihs

…but `scipy.linalg` can be used to analyse stability for LTI systems

Self-assessment material

Question 1

Does the concept of marginal stability of an equilibrium apply only to LTI systems?

Potential answers:

I: Yes, marginal stability is defined only for LTI systems.
II: No, marginal stability can be defined for nonlinear systems as well.
III: Marginal stability is irrelevant for LTI systems.
IV: It only applies to mechanical systems.
V: I do not know

Question 2

Does the concept of marginal stability of an equilibrium apply only to continuous-time systems?

Potential answers:

- I: Yes, marginal stability is only defined for continuous-time systems.
- II: No, but it is more relevant in continuous-time systems.
- III: No, discrete-time systems do not have equilibria.
- IV: No, marginal stability can be defined for both continuous and discrete-time systems.
- V: I do not know

Question 3

In the game of marginal stability, who starts? The boss or the apprentice?

Potential answers:

I: The apprentice, since they test small perturbations.
II: The boss, since the system dynamics dictate the response.
III: They both start at the same time.
IV: There is no turn-based order in stability analysis.
V: I do not know

Question 4

If a system has a marginally stable equilibrium, then all its equilibria must be marginally stable. Is this statement correct?

Potential answers:

I: No, stability properties are equilibrium-dependent.
II: Yes, if one equilibrium is marginally stable, all others must be as well.
III: The question is meaningless because marginal stability does not exist.
IV: Only if the system is conservative.
V: I do not know

Question 5

Is the origin for the Lotka-Volterra model simply stable?

Potential answers:

I: No, it is a saddle point and therefore unstable.
II: Yes, because populations always return to equilibrium.
III: Yes, because it has only non-positive eigenvalues.
IV: It depends on the initial conditions.
V: I do not know

Recap of sub-module “explain and determine the marginal stability of an equilibrium”

- marginal stability / simple stability is the property that answers the question “can I bound the evolutions, i.e., arbitrarily constrain them to do not get “too far” from an equilibrium by starting opportunely closeby the original equilibrium?
- an equilibrium is marginally stable or not depending on whether one is able to ‘win’ the ‘choose your neighborhood’ game
- phase portraits are very interpretable, to this regards
- there is a sort of “downgrading” phenomenon that happens here: one has to have *all* the trajectories behaving in a good way to have a certain property. One not behaving is enough for the “downgrading” of the equilibrium

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