computing free evolutions and forced responses of LTI systems

Contents map

developed content units	taxonomy levels
free evolution	u1, e1
forced response	u1, e1

prerequisite content units	taxonomy levels
LTI RR	u1, e1
convolution	u1, e1
partial fraction decomposition	u1, e1

Main ILO of sub-module "computing free evolutions and forced responses of LTI systems"

Compute free evolutions and forced responses of LTI systems using Z-transforms based formulas (but only as procedural tools)

Disclaimer

the formulas introduced in this module shall be taken as "ex machina"

Focus in this module = on ARMA models

$$y^{[n]} = a_{n-1} y^{[n-1]} + \ldots + a_0 y + b_m u^{[m]} + \ldots + b_0 u$$

with $^{[i]}$ meaning the i-th step ahead sample (e.g, $y^{[3]} = y^{+++}$).

Focus in this module = on ARMA models

$$y^{[n]} = a_{n-1}y^{[n-1]} + \ldots + a_0 y + b_m u^{[m]} + \ldots + b_0 u$$

with $^{[i]}$ meaning the i-th step ahead sample (e.g, $y^{[3]} = y^{+++}$). *Discussion:* why is the LHS $y^{[n]}$ and not $a_n y^{[n]}$?

Focus in this module = on ARMA models

$$y^{[n]} = a_{n-1} y^{[n-1]} + \ldots + a_0 y + b_m u^{[m]} + \ldots + b_0 u$$

with $^{[i]}$ meaning the i-th step ahead sample (e.g, $y^{[3]} = y^{+++}$). *Discussion:* why is the LHS $y^{[n]}$ and not $a_n y^{[n]}$? *Discussion:* and which initial conditions shall we consider?

Z transforms - links for who would like to get more info about them

Z transforms = discretization of Laplace transforms; interesting material:

- https://www.youtube.com/watch?v=XJRW6jamUHk
- https://www.youtube.com/watch?v=acQecd6dmxw
- https://www.youtube.com/watch?v=4PV6ikgBShw
- https://www.youtube.com/watch?v=7Gl4kJUjp4c

Laplace transforms - links for who would like to get more info about them

Laplace transforms = extension of Fourier transforms; interesting material:

- `https://www.youtube.com/watch?v=r6sGWTCMz2k` (Fourier series)
- `https://www.youtube.com/watch?v=spUNpyF58BY` (Fourier transforms)
- `https://www.youtube.com/watch?v=nmgFG7PUHfo` (on the historical importance of Fast Fourier Transforms)
- `https://www.youtube.com/watch?v=7UvtU75NXTg` (Laplace Transforms, in math)
- `https://www.youtube.com/watch?v=n2y7n6jw5d0` (Laplace Transforms, graphically)

Main usefulness: convolution in time transforms into multiplication in Z-domain, and viceversa

$$\begin{cases} H(z) = \mathcal{Z}\{h[k]\} \\ U(z) = \mathcal{Z}\{u[k]\} \end{cases} \implies \mathcal{Z}\{h * u[k]\} = H(z)U(z)$$

Main usefulness: convolution in time transforms into multiplication in Z-domain, and viceversa

$$\begin{cases} H(z) = \mathcal{Z}\{h[k]\} \\ U(z) = \mathcal{Z}\{u[k]\} \end{cases} \implies \mathcal{Z}\{h * u[k]\} = H(z)U(z)$$

Noticeable name: *transfer function* $\left(= H(z) = \mathcal{Z}\{\text{impulse response}\}\right)$

An intuitive explanation of the usefulness of the Zeta transform in automatic control

initial value problem

$$\boxed{x + 0.3x^{-} = 0, \quad x[-1] = 2}$$

An intuitive explanation of the usefulness of the Zeta transform in automatic control

initial value problem

$$x + 0.3x^{-} = 0, \quad x[-1] = 2$$

$$x[k] = 2(-0.3)^{k}$$

solution in the time domain

An intuitive explanation of the usefulness of the Zeta transform in automatic control

initial value problem

$$x + 0.3x^{-} = 0, \quad x[-1] = 2$$

$\xrightarrow{\mathcal{Z}}$

algebraic problem

$$X(z) + 0.3z^{-1}X(z) + x[-1] = 0$$

$$x[k] = 2(-0.3)^{k}$$

solution in the time domain

An intuitive explanation of the usefulness of the Zeta transform in automatic control

initial value problem

$$x + 0.3x^{-} = 0, \quad x[-1] = 2$$

$\xrightarrow{\mathcal{Z}}$

algebraic problem

$$X(z) + 0.3z^{-1}X(z) + x[-1] = 0$$

algebra

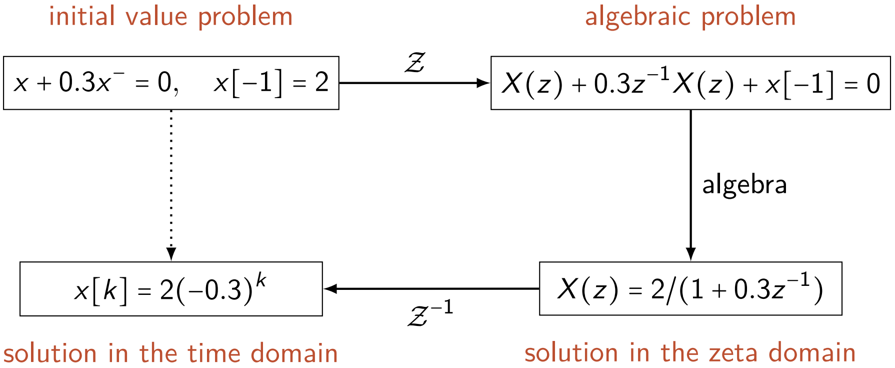
$$X(z) = 2/(1 + 0.3z^{-1})$$

solution in the zeta domain

$$x[k] = 2(-0.3)^{k}$$

solution in the time domain

An intuitive explanation of the usefulness of the Zeta transform in automatic control



First set of formulas to memorize: Zeta-transforming derivatives

(these will be motivated in other courses)

$$\mathcal{Z}\left\{x^{-}\right\} = z^{-1}X(z) + x[-1]$$

First set of formulas to memorize: Zeta-transforming derivatives

(these will be motivated in other courses)

$$\mathcal{Z}\left\{x^{-}\right\} = z^{-1}X(z) + x[-1]$$

$$\mathcal{Z}\left\{x^{--}\right\} = z^{-2}X(z) + x[-2] + z^{-1}x[-1]$$

First set of formulas to memorize: Zeta-transforming derivatives

(these will be motivated in other courses)

$$\mathcal{Z}\left\{x^{-}\right\} = z^{-1}X(z) + x[-1]$$

$$\mathcal{Z}\left\{x^{--}\right\} = z^{-2}X(z) + x[-2] + z^{-1}x[-1]$$

$$\mathcal{Z}\left\{x^{---}\right\} = z^{-3}X(z) + x[-3] + z^{-1}x[-2] + z^{-2}x[-1]$$

First set of formulas to memorize: Zeta-transforming derivatives

(these will be motivated in other courses)

$$\mathcal{Z}\left\{x^{-}\right\} = z^{-1}X(z) + x[-1]$$

$$\mathcal{Z}\left\{x^{--}\right\} = z^{-2}X(z) + x[-2] + z^{-1}x[-1]$$

$$\mathcal{Z}\left\{x^{---}\right\} = z^{-3}X(z) + x[-3] + z^{-1}x[-2] + z^{-2}x[-1]$$

$$\mathcal{Z}\left\{x^{[-m]}\right\} = \dots$$

Example: discretized spring mass system

$$y + \alpha_1 y^{-} + \alpha_2 y^{--} + \beta u^{--}$$

$$\Downarrow$$

$$Y(z) + \left(\alpha_1 z^{-1} Y(z) + \alpha_1 y[-1]\right) + \left(\alpha_2 z^{-2} Y(z) + \alpha_2 y[-2] + \alpha_2 z^{-1} y[-1]\right) = \beta z^{-2} U(z)$$

$$\Downarrow$$

$$Y(z) + \alpha_1 z^{-1} Y(z) + \alpha_2 z^{-2} Y(z) = \alpha_1 y[-1] + \alpha_2 y[-2] + \alpha_2 z^{-1} y[-1] + \beta z^{-2} U(z)$$

$$\Downarrow$$

$$Y(z) = \frac{\alpha_1 y[-1] + \alpha_2 y[-2] + \alpha_2 z^{-1} y[-1]}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}} + \frac{z^{-2}\beta}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}} U(z)$$

And what shall we do once we get this?

generalizing the previous slide: $Y(z) = \frac{M(z)}{A(z)} + \frac{B(z)}{A(z)} U(z)$

with

- $\frac{M(z)}{A(z)}$ = Zeta transform of the free evolution
- $\frac{B(z)}{A(z)} U(z)$ = Zeta transform of the forced response

$\Longrightarrow$ we shall anti-transform; how?

And what shall we do once we get this?

generalizing the previous slide: $Y(z) = \frac{M(z)}{A(z)} + \frac{B(z)}{A(z)} U(z)$

with

- $\frac{M(z)}{A(z)}$ = Zeta transform of the free evolution
- $\frac{B(z)}{A(z)} U(z)$ = Zeta transform of the forced response

$\Longrightarrow$ we shall anti-transform; how? Main 2 cases:

- either $U(z) = \frac{\text{polynomial in } z}{\text{polynomial in } z}$
- or $U(z)$ = something else

first case: rational $U(z)$

How to do if $U(z) = \dfrac{\text{polynomial in } z}{\text{polynomial in } z}$

$$Y(z) = \frac{M(z)}{A(z)} + \frac{B(z)}{A(z)} U(z) \quad \mapsto \quad Y(z) = \frac{M(z)}{A(z)} + \frac{C(z)}{D(z)}$$

How to do if $U(z) = \dfrac{\text{polynomial in } z}{\text{polynomial in } z}$

$$Y(z) = \frac{M(z)}{A(z)} + \frac{B(z)}{A(z)} U(z) \quad \mapsto \quad Y(z) = \frac{M(z)}{A(z)} + \frac{C(z)}{D(z)}$$

write each of the two parts of the signal as

$$\frac{N(z)}{(z - \lambda_1)(z - \lambda_2)(z - \lambda_3) \cdots}$$

Next step: partial fraction decomposition

- **case single poles:** if $\dfrac{N(z)}{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)\cdots}$ is s.t. $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots$ then there exist $\alpha_1, \alpha_2, \alpha_3, \ldots$ s.t.

$$\frac{N(z)}{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)\cdots} = \frac{\alpha_1 z}{z-\lambda_1} + \frac{\alpha_2 z}{z-\lambda_2} + \frac{\alpha_3 z}{z-\lambda_3} + \cdots \tag{1}$$

Next step: partial fraction decomposition

- **case single poles:** if $\frac{N(z)}{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)\cdots}$ is s.t. $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots$ then there exist $\alpha_1, \alpha_2, \alpha_3, \ldots$ s.t.
$$\frac{N(z)}{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)\cdots} = \frac{\alpha_1 z}{z-\lambda_1} + \frac{\alpha_2 z}{z-\lambda_2} + \frac{\alpha_3 z}{z-\lambda_3} + \cdots \tag{1}$$

- **case repeated poles:** if some poles are repeated, then there exist $\alpha_{1,1}, \ldots,$ $\alpha_{1,n1}, \alpha_{2,1}, \ldots, \alpha_{2,n2}, \ldots,$ s.t.
$$\frac{N(z)}{(z-\lambda_1)^{n1}(z-\lambda_2)^{n2}\ldots} = \frac{\alpha_{1,1}(z)}{z-\lambda_1} + \ldots + \frac{\alpha_{1,n1}(z)}{(z-\lambda_1)^{n1}} + \frac{\alpha_{2,1}(z)}{z-\lambda_2} + \ldots + \frac{\alpha_{2,n2}(z)}{(z-\lambda_2)^{n2}} + \ldots \tag{2}$$

Next step: partial fraction decomposition

- **case single poles:** if $\dfrac{N(z)}{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)\cdots}$ is s.t. $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots$ then there exist $\alpha_1, \alpha_2, \alpha_3, \ldots$ s.t.

$$\frac{N(z)}{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)\cdots} = \frac{\alpha_1 z}{z-\lambda_1} + \frac{\alpha_2 z}{z-\lambda_2} + \frac{\alpha_3 z}{z-\lambda_3} + \cdots \tag{1}$$

- **case repeated poles:** if some poles are repeated, then there exist $\alpha_{1,1}, \ldots, \alpha_{1,n1}, \alpha_{2,1}, \ldots, \alpha_{2,n2}, \ldots,$ s.t.

$$\frac{N(z)}{(z-\lambda_1)^{n1}(z-\lambda_2)^{n2}\ldots} = \frac{\alpha_{1,1}(z)}{z-\lambda_1} + \ldots + \frac{\alpha_{1,n1}(z)}{(z-\lambda_1)^{n1}} + \frac{\alpha_{2,1}(z)}{z-\lambda_2} + \ldots + \frac{\alpha_{2,n2}(z)}{(z-\lambda_2)^{n2}} + \ldots \tag{2}$$

"But how do I compute α_1, α_2, etc.?" $\mapsto$
`en.wikipedia.org/wiki/Partial_fraction_decomposition`
(tip: start from `en.wikipedia.org/wiki/Heaviside_cover-up_method`)

Anti-transforming in the rational $U(z)$ and simple poles case

$$\text{if } Y(z) = \frac{\alpha_1 z}{z - \lambda_1} + \frac{\alpha_2 z}{z - \lambda_2} + \ldots \text{ then use}$$

$$\mathcal{Z}\left\{\alpha\lambda^k\right\} = \frac{\alpha z}{z - \lambda} \quad \leftrightarrow \quad \mathcal{Z}^{-1}\left\{\frac{\alpha z}{z - \lambda}\right\} = \alpha\lambda^k$$

Anti-transforming in the rational $U(z)$ case

$$\text{if } Y(z) = \frac{\alpha_{1,1} z}{z - \lambda_1} + \ldots + \frac{\alpha_{1,n1}(z)}{(z - \lambda_1)^{n1}} + \frac{\alpha_{2,1} z}{z - \lambda_2} + \ldots + \frac{\alpha_{2,n2}(z)}{(z - \lambda_2)^{n2}} + \ldots$$

then use some software suite!

Something though to remember

$$\mathcal{Z}\left\{k^m \lambda^k\right\} \propto \frac{\star(z)}{(z-\lambda)^{m+1}}$$

Numerical Example: Inverse Zeta Transform of a Rational Function

$$Y(z) = \frac{3z}{z-2} + \frac{5z}{z+1}$$

goal = compute the inverse Zeta transform $y[k] = \mathcal{Z}^{-1}\{Y(z)\}$

Step 1: Identify the terms

$$Y(z) = \frac{3z}{z-2} + \frac{5z}{z+1}$$

Here:

- $\lambda_1 = 2$, with coefficient $\alpha_{1,1} = 3$
- $\lambda_2 = -1$, with coefficient $\alpha_{2,1} = 5$

Step 2: Apply the inverse Zeta transform formula

by means of

$$\mathcal{Z}^{-1}\left\{\frac{z}{(z-\lambda)}\right\} = \lambda^k$$

we compute the inverse Zeta transform of each term:

- $\mathcal{Z}^{-1}\left\{\frac{3z}{z-2}\right\} = 3 \cdot 2^k$
- $\mathcal{Z}^{-1}\left\{\frac{5z}{z+1}\right\} = 5 \cdot (-1)^k$

Step 3: Combine the results

then we have that the inverse Zeta transform $y[k]$ is the sum of the individual transforms, i.e.,

$$y[k] = 3 \cdot 2^k + 5 \cdot (-1)^k$$

Another thing to remember: complex conjugate poles lead to sinusoidal modes

$$\mathcal{Z}\left\{\cos(\omega k)\right\} = \frac{z\big(z - \cos(\omega)\big)}{z^2 - 2z\cos(\omega) + 1}$$

Extremely important result

a LTI in free evolution behaves as a combination of terms λ^k, $k\lambda^k$, $k^2\lambda^k$, etc. for a set of different λ's and powers of k, called the *modes* of the system

Extremely important result

a LTI in free evolution behaves as a combination of terms λ^k, $k\lambda^k$, $k^2\lambda^k$, etc. for a set of different λ's and powers of k, called the *modes* of the system

Discussion: assuming that we have two modes, $(0.3)^k$ and $(-0.9)^k$, so that

$$y[k] = \alpha_1 0.3^k + \alpha_2(-0.9)^k.$$

What determines α_1 and α_2?

second case: irrational $U(z)$

In this case we cannot use partial fractions decompositions as before

$$\text{from } Y(z) = \frac{M(z)}{A(z)} + \frac{B(z)}{A(z)} U(z) \text{ we follow the algorithm}$$

- find $y_{\text{free}}[k]$ from PFDs of $\frac{M(z)}{A(z)}$ as before
- find the impulse response $h[k]$ from PFDs of $\frac{B(z)}{A(z)}$ as before
- find $y_{\text{forced}}[k]$ as $h * u[k]$

Summarizing

Compute free evolutions and forced responses of LTI systems using Z-transforms based formulas (but only as procedural tools)

- Z-transform the DT ARMA
- if $u[k]$ admits a rational $U(z)$ then write $Y(z) = \frac{\text{polynomial}}{\text{polynomial}}$, do PFD, and do inverse-Zetas
- if $u[k]$ does not admit a rational $U(z)$, do similarly as before but do PFD only for the free evolution and impulse response, and find the forced response by means of convolution

Most important python code for this sub-module

Two essential libraries

- `https://python-control.readthedocs.io/en/0.10.1/generated/control.modal_form.html`
- `https://docs.sympy.org/latest/modules/physics/control/lti.html`

Self-assessment material

Question 1

What is the primary purpose of using Z-transforms in the context of LTI systems?

Potential answers:

I: To convert convolution in the time domain into multiplication in the Z-domain.
II: To derive the Laplace transform from the Fourier transform.
III: To compute the eigenvalues of the system matrix.
IV: To solve partial differential equations directly.
V: I do not know

Question 2

In the ARMA model $y^{[n]} = a_{n-1} y^{[n-1]} + \ldots + a_0 y + b_m u^{[m]} + \ldots + b_0 u$, why is the left-hand side $y^{[n]}$ and not $a_n y^{[n]}$?

Potential answers:

I: To work with monic polynomials, simplifying the analysis.
II: To ensure the system is always stable.
III: To make the system nonlinear.
IV: To reduce the number of initial conditions required.
V: I do not know

Question 3

What is the purpose of partial fraction decomposition in the context of Z-transforms?

Potential answers:

I: To break down a complex rational function into simpler terms for inverse Z-transform.
II: To compute the convolution of two signals directly.
III: To derive the Laplace transform from the Z-transform.
IV: To solve nonlinear differential equations.
V: I do not know

Question 4

What are the modes of a LTI system in free evolution?

Potential answers:

I: Combinations of terms like λ^k, $k\lambda^k$, $k^2\lambda^k$, etc.

II: The eigenvalues of the system matrix.

III: The coefficients of the ARMA model.

IV: The initial conditions of the system.

V: I do not know

Question 5

How is the forced response of a LTI system computed when $U(z)$ is not rational?

Potential answers:

I: By computing the convolution of the impulse response $h[k]$ with the input $u[k]$.

II: By using partial fraction decomposition on $U(z)$.

III: By directly inverting the Z-transform of $U(z)$.

IV: By solving the system's differential equations numerically.

V: I do not know

Recap of sub-module "computing free evolutions and forced responses of LTI systems"

- finding such signals require knowing a couple of formulas by heart
- partial fraction decomposition is king here, one needs to know how to do that

?