1D convolution in discrete time

Contents map

developed content units	**taxonomy levels**
convolution	u1, e1

prerequisite content units	**taxonomy levels**
signal	u1, e1

Main ILO of sub-module "1D convolution in discrete time"

Compute the convolution between two single dimensional discrete time signals

Why convolution?

because for a LTI system with impulse response $h[k]$
it follows that $y_{\text{forced}}[k] = u * h[k]$

extremely important result for LTI systems:

$$y_{\text{forced}}[k] = h * u[k] = u * h[k] :=$$

$$:= \sum_{\kappa=-\infty}^{+\infty} u[\kappa] h[k-\kappa] = \sum_{-\infty}^{+\infty} h[\kappa] u[k-\kappa]$$

... and this module = what that formula actually means from graphical perspectives

Additional material

Videos:

- https://www.youtube.com/watch?v=KuXjwB4LzSA
- https://www.youtube.com/watch?v=acAw5WGtzuk
- https://www.youtube.com/watch?v=IaSGqQa5O-M (for connections with probability)
- https://www.youtube.com/playlist?list=PL4iThgVpN7hmbIhHnCa7SDOOgLMoNwED_
- https://www.youtube.com/playlist?list=PL4mJLdGEHNvhCuPXsKFrnD7AaQBlMEB6a

Animations:

- https://lpsa.swarthmore.edu/Convolution/CI.html
- https://phiresky.github.io/convolution-demo/

Towards decomposing this formula in pieces

$$y_{\text{forced}}[k] = h * u[k] = \sum_{-\infty}^{+\infty} u[\kappa]h[k-\kappa] = \sum_{-\infty}^{+\infty} h[\kappa]u[k-\kappa]$$

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$$y_{\text{forced}}[k] = h * u[k] = \sum_{-\infty}^{+\infty} u[\kappa]h[k-\kappa] = \sum_{-\infty}^{+\infty} h[\kappa]u[k-\kappa]$$

better focus on

$$\sum_{-\infty}^{+\infty} u[\kappa]h[k-\kappa]$$

or on

$$\sum_{-\infty}^{+\infty} h[\kappa]u[k-\kappa] \; ?$$

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or on

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in automatic control typically better the second

Towards decomposing this formula in pieces, small change of notation

$$y_{\text{forced}}[k] = \sum_{-\infty}^{+\infty} h[\kappa] u[k - \kappa] \quad \mapsto \quad y_{\text{forced}}[\mathrm{now}] = \sum_{-\infty}^{+\infty} h[\kappa] u[\mathrm{now} - \kappa]$$

Decomposing this formula in pieces

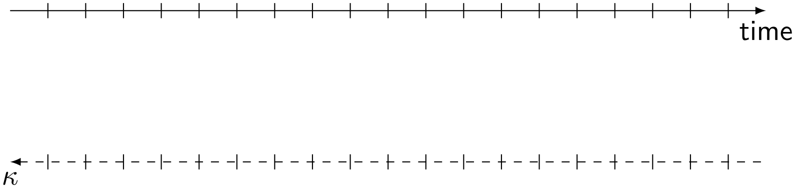
$$y_{\text{forced}}[\mathrm{now}] = \sum_{-\infty}^{+\infty} h[\kappa]u[\mathrm{now} - \kappa]$$

$\Longrightarrow$ constituent pieces =

- $u[\mathrm{now} - \kappa]$
- $h[\kappa]$
- $u[\mathrm{now} - \kappa]h[\kappa]$
- $\sum u[\mathrm{now} - \kappa]h[\kappa]$

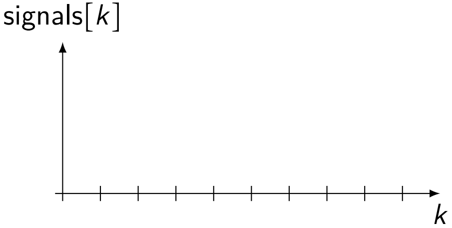
Visualizing the various pieces

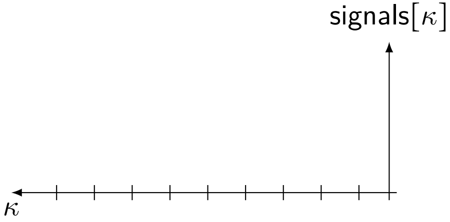
$$u[\mathrm{now} - \kappa] \qquad h[\kappa] \qquad u[\mathrm{now} - \kappa]h[\kappa] \qquad \sum_{-\infty}^{+\infty} h[\kappa]u[\mathrm{now} - \kappa]$$



Example

$$u[k] = \begin{cases} 1 & \text{for } k \in \{1, 2\} \\ 0 & \text{otherwise} \end{cases} \qquad h[k] = \begin{cases} 2 & \text{for } k \in \{0, 1\} \\ 1 & \text{for } k \in \{2\} \\ 0 & \text{otherwise} \end{cases}$$



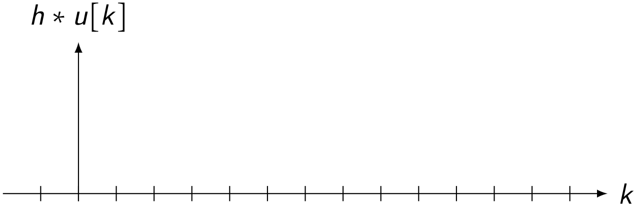


Another Example:

$$u[k] = \begin{cases} 1 & \text{for } k \in \{0, 1, 3\} \\ 0 & \text{otherwise} \end{cases} \qquad h[k] = \begin{cases} 2 & \text{for } k \in \{0, 1, 2\} \\ 0 & \text{otherwise} \end{cases} \qquad ?$$

Paramount message

h in $y_{\text{forced}}[k] = \sum_{-\infty}^{+\infty} h[\kappa]u[k-\kappa]$ represents how much the past u's contribute to the current y_{forced}:



Refreshing what we are doing and why

Dynamics of a cart: $v[k+1] = \alpha v[k] + \beta F[k]$ with:

- **control input:** $u[k]$ (actuation from the motor, in this case = $F[k]$)

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- **forced response:** $y_{\text{forced}}[k] = u * h[k]$ (output in time corresponding to null initial condition, i.e., $y[0] = 0$, and input $u[k]$ whatever it is)
- **total response:** $y[k] = y_{\text{free}}[k] + y_{\text{forced}}[k]$

Quiz time!

$$h * u[k] := \sum_{-\infty}^{+\infty} h[\kappa]u[k-\kappa]$$

- is $h * u[k] = u * h[k]$?

Quiz time!

$$h * u[k] := \sum_{-\infty}^{+\infty} h[\kappa] u[k - \kappa]$$

- is $h * u[k] = u * h[k]$?
- is $(\alpha h_1 + \beta h_2) * u[k] = \alpha (h_1 * u[k]) + \beta (h_2 * u[k])$?

Quiz time!

$$h * u[k] := \sum_{-\infty}^{+\infty} h[\kappa]u[k-\kappa]$$

- is $h * u[k] = u * h[k]$?
- is $(\alpha h_1 + \beta h_2) * u[k] = \alpha\left(h_1 * u[k]\right) + \beta\left(h_2 * u[k]\right)$?
- if both $h[\kappa] = 0$ and $u[k] = 0$ if $t < 0$, how can we simplify $y[k] = \sum_{-\infty}^{+\infty} h[\kappa]u[k-\kappa]$?

Summarizing

Compute the convolution between two single dimensional continuous time signals

- take one of the two signals
- translate it to the "current k"
- flip it
- multiply the two signals in a pointwise fashion
- compute the discrete integral of the result

Most important python code for this sub-module

Methods implementing (discrete) convolutions

- https://numpy.org/doc/2.1/reference/generated/numpy.convolve.html
- https://docs.scipy.org/doc/scipy/reference/generated/scipy.signal.convolve.html

Self-assessment material

Question 1

What does the convolution integral $y_{\text{forced}}[k] = \sum_{-\infty}^{+\infty} h[\kappa]u[k-\kappa]$ represent in the context of LTI systems?

Potential answers:

I: The free evolution of the system output.
II: The forced response of the system output due to the input $u[k]$.
III: The total response of the system, including initial conditions.
IV: The impulse response of the system.
V: I do not know.

Question 2

Which of the following is true about the convolution operation $h * u[k]$?

Potential answers:

I: It is only defined for periodic signals.
II: It is only applicable to discrete-time systems.
III: It is commutative, i.e., $h * u[k] = u * h[k]$.
IV: It requires both signals to be symmetric.
V: I do not know.

Question 3

What does the impulse response $h[k]$ of an LTI system represent?

Potential answers:

I: The input signal $u[k]$ applied to the system.
II: The free evolution of the system output.
III: The total response of the system, including initial conditions.
IV: The output of the system when the input is a Dirac delta function $\delta[k]$.
V: I do not know.

Question 4

If $h[\kappa] = 0$ for $\kappa < 0$ and $u[k] = 0$ for $t < 0$, how can the convolution integral $y[k] = \sum_{-\infty}^{+\infty} h[\kappa]u[k-\kappa]$ be simplified?

Potential answers:

I: $y[k] = \sum_{0}^{t} h[\kappa]u[k-\kappa]$

II: $y[k] = \sum_{0}^{+\infty} h[\kappa]u[k-\kappa]$

III: $y[k] = \sum_{-\infty}^{+\infty} h[\kappa]u[\kappa]$

IV: $y[k] = \sum_{-\infty}^{0} h[\kappa]u[k-\kappa]$

V: I do not know.

Question 5

What is the graphical interpretation of $h[\kappa]$ in the convolution integral $y_{\text{forced}}[k] = \sum_{-\infty}^{+\infty} h[\kappa]u[k-\kappa]$?

Potential answers:

I: It represents the future inputs of the system.
II: It represents how much past inputs contribute to the current output.
III: It represents the free evolution of the system.
IV: It represents the total energy of the system.
V: I do not know.

Recap of sub-module "1D convolution in discrete time"

- convolution is an essential operator, since it can be used for LTI systems to compute forced responses
- its graphical interpretation aids interpreting impulse responses as how the past inputs contribute to current outputs

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