what is the superposition principle, and what does it imply

Contents map

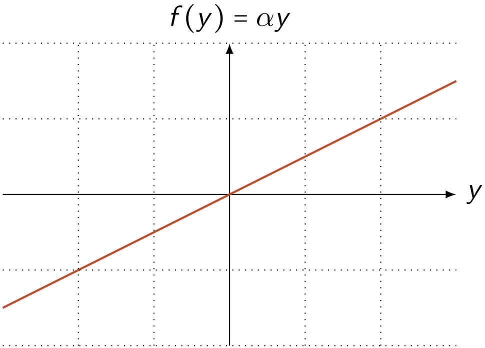
developed content units	taxonomy levels
superposition principle	u1, e1

prerequisite content units	taxonomy levels
LTI RR	u1, e1

Main ILO of sub-module
“what is the superposition principle, and what does it imply”

Describe the importance of the super-
position principle to analyze LTI systems

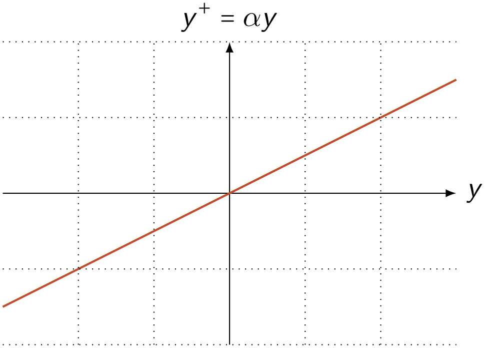
Starting with graphs



implications/definition of linearity:

- $f(x + y) = f(x) + f(y)$
- $f(\alpha y) = \alpha f(y)$

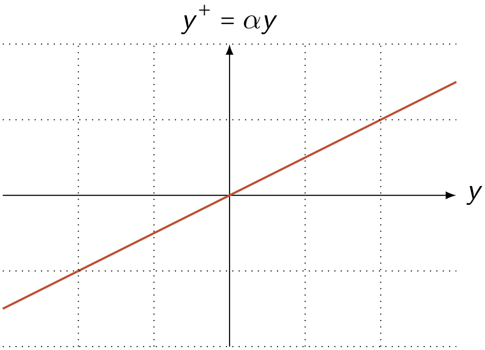
What if we interpret this as a RR?



$\implies$ an LTI system, for which

$$y^+ = \alpha y \quad \text{is solved by} \quad y[k] = y[0]\alpha^k \quad \forall y[0], \alpha, k$$

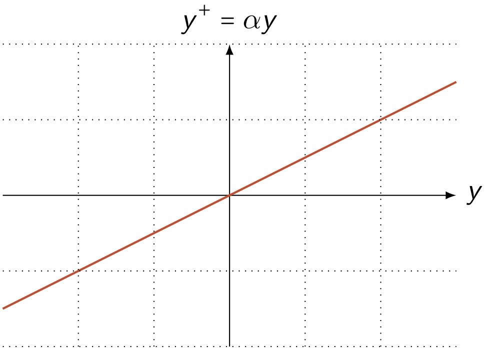
And can we build on top of this?



$\implies$ an LTI system, for which

- $y'[0] = 2 \mapsto y'[k] = 2\alpha^k$

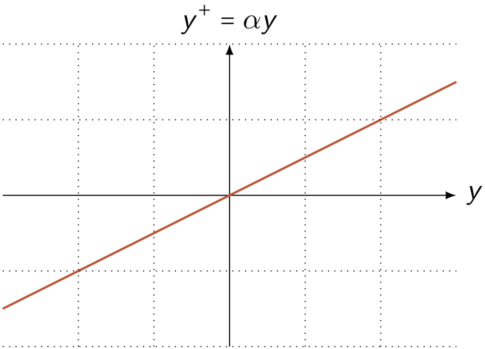
And can we build on top of this?



$\implies$ an LTI system, for which

- $y'[0] = 2 \mapsto y'[k] = 2\alpha^k$
- $y''[0] = 3 \mapsto y''[k] = 3\alpha^k$

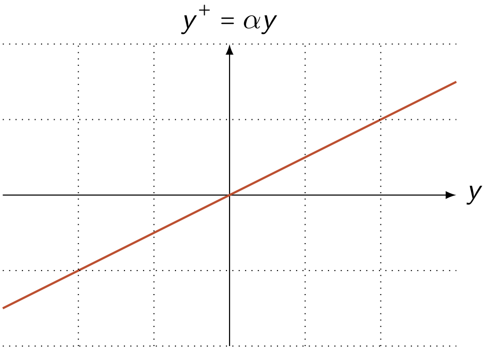
And can we build on top of this?



$\implies$ an LTI system, for which

- $y'[0] = 2 \mapsto y'[k] = 2\alpha^k$
- $y''[0] = 3 \mapsto y''[k] = 3\alpha^k$
- $y'''[0] = 3 + 2 \mapsto y'''[k] = (3+2)\alpha^k$

And can we build on top of this?



$\implies$ an LTI system, for which

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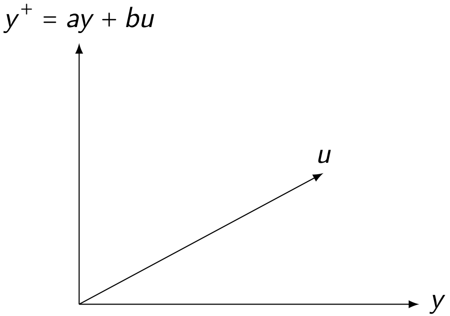
- $y'''[0] = 3 + 2 \mapsto y'''[k] = (3 + 2)\alpha^k$

$$y'[0] \;+\; y''[0] \quad \mapsto \quad y'[k] \;+\; y''[k]$$

Further generalization



- $\left\{y'[0], u'\right\} \mapsto y'[k]$
- $\left\{y''[0], u''\right\} \mapsto y''[k]$
- $\left\{y'[0] + y''[0], u' + u''\right\} \mapsto y'[k] + y''[k]$

Aiding intuitions with math

Linearity implies that if $\{y', u', y'[0]\}$ and $\{y'', u'', y''[0]\}$ satisfy

$$\begin{cases} y'[k+1] & = ay'[k] + bu'[k] \\ y'[0] & = y'_0 \\ \\ y''[k+1] & = ay''[k] + bu''[k] \\ y''[0] & = y''_0 \end{cases} \tag{1}$$

then their sum also satisfies

$$\begin{cases} \big(\alpha' y'[k+1] + \alpha'' y''[k+1]\big) & = a\big(\alpha' y'[k] + \alpha'' y''[k]\big) + b\big(\alpha' u'[k] + \alpha'' u''[k]\big) \\ \alpha' y'[0] + \alpha'' y''[0] & = \alpha' y'_0 + \alpha'' y''_0 \end{cases} \tag{2}$$

Rephrasing

Linearity implies that if $\{y', u', y'[0]\}$ and $\{y'', u'', y''[0]\}$ satisfy the RR then also their sum $\{y' + y'', u' + u'', y'[0] + y''[0]\}$ satisfies the RR.

Rephrasing

Linearity implies that if $\{y', u', y'[0]\}$ and $\{y'', u'', y''[0]\}$ satisfy the RR then also their sum $\{y' + y'', u' + u'', y'[0] + y''[0]\}$ satisfies the RR.

The superposition principle in words

in LTI systems
combining inputs and initial conditions
produces a total effect
that is the linear combination
of that effects
one would get with the individual causes
each acting separately

Important: the superposition principle works with any LTI

Will be repeated and stated again precisely later on

the proof holds for every system that generalizes $y^{+} = ay + bu$,
i.e., every "linear combination of temporal shifts of y = linear combination of temporal shifts of u"

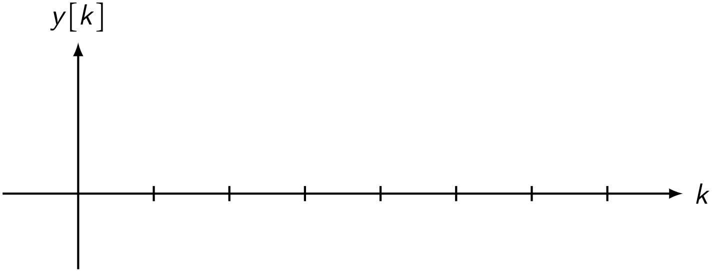
Superposition principle $\Longrightarrow$

response of LTIs $=$ free evolution $+$ forced response

Superposition principle $\implies$ response of LTIs = free evolution + forced response

assume:

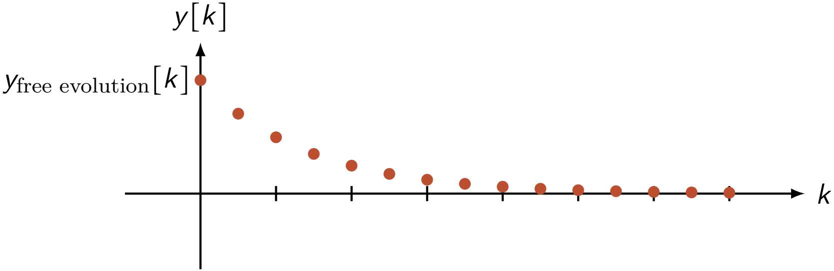
- $y^+ = ay + bu$
-
-



Superposition principle $\Longrightarrow$
response of LTIs = free evolution + forced response

assume:

- $y^+ = ay + bu$
- $\{u[k] = 0[k], \quad y[0] \neq 0\}$ causes $y_{\text{free evolution}}[k]$
-



Superposition principle $\Longrightarrow$

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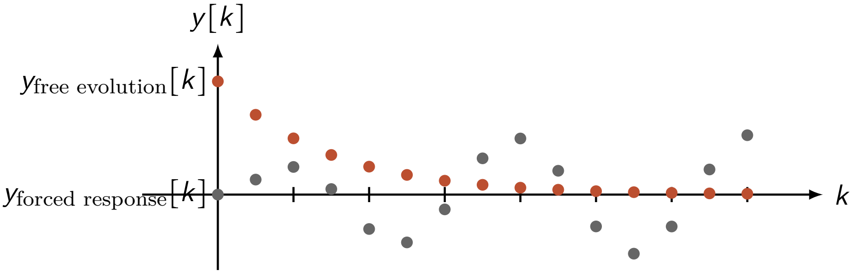
assume:

- $y^+ = ay + bu$
- $\{u[k] = 0[k], \quad y[0] \neq 0\}$ causes $y_{\text{free evolution}}[k]$
- $\{u[k] \neq 0[k], \quad y[0] = 0\}$ causes $y_{\text{forced response}}[k]$



Superposition principle $\implies$

response of LTIs = free evolution + forced response

assume:

- $y^+ = ay + bu$
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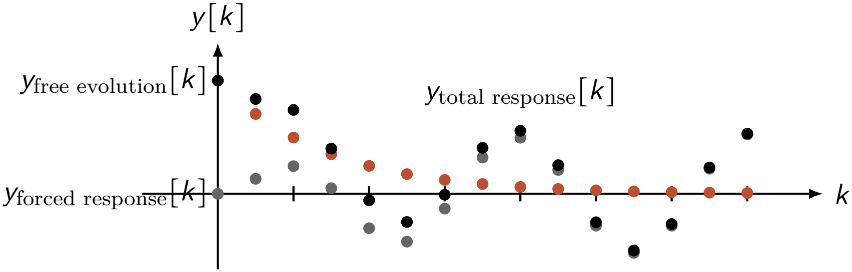


then $\{u[k] \neq 0[k], y[0] \neq 0\}$ causes $y_{\text{free evolution}}[k] + y_{\text{forced response}}[k]$

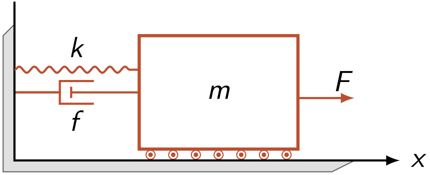
A mnemonic scheme

(only for LTI systems!!)

$$(u, y_0) = (0, y_0) + (u, 0)$$

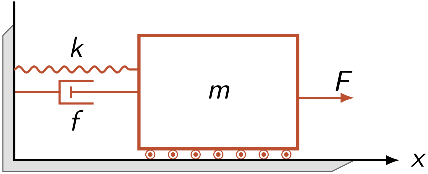
total response = free evolution + forced response

Continuing with some intuitions



Discussion: how will the cart move if I use $u[k] = \sin(\omega k T)$ starting from a resting state? (only intuitively, assuming everything ideal)

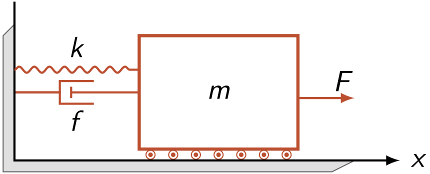
Continuing with some intuitions



Discussion: how will the cart move if I use $u[k] = \sin(\omega k T)$ starting from a resting state? (only intuitively, assuming everything ideal)

And what about if $u[k] = 2\sin(\omega k T)$?

Continuing with some intuitions



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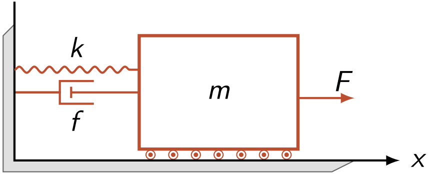
Continuing with some intuitions



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And what about if $u[k] = 2\sin(\omega k T)$?

And what about $u[k] = \sin\left(\omega' k T\right) + \sin\left(\omega'' k T\right)$?

And what about $u[k] = \alpha' \sin\left(\omega' k T\right) + \alpha'' \sin\left(\omega'' k T\right)$?

Summarizing

Describe the importance of the superposition principle to analyze LTI systems

- it makes us able to say "total = free + forced"

Most important python code for this sub-module

Suggestion

part of the SciPy library (`scipy.signal`) provides tools for working with LTI systems, including creating transfer functions, state-space representations, and analyzing system responses (stuff that will be seen in the next modules)

Self-assessment material

Question 1

What is the primary implication of the superposition principle in LTI systems?

Potential answers:

I: The total response is the sum of the free evolution and the forced response.
II: The system response is always exponential.
III: The system response is independent of the initial conditions.
IV: The system response is nonlinear.
V: I do not know

Question 2

Which of the following properties is essential for a system to be considered linear?

Potential answers:

I: The system response is always sinusoidal.

II: The system satisfies the properties $f(x+y) = f(x)+f(y)$ and $f(\alpha y) = \alpha f(y)$.

III: The system response is independent of the input.

IV: The system response is always zero for zero input.

V: I do not know

Question 3

What happens to the response of an LTI system if the input is scaled by a factor α?

Potential answers:

I: The response becomes nonlinear.
II: The response remains unchanged.
III: The response is scaled by the same factor α.
IV: The response becomes zero.
V: I do not know

Question 4

What is the significance of the superposition principle in analyzing LTI systems?

Potential answers:

I: It allows us to ignore the initial conditions.

II: It allows us to decompose the system response into free evolution and forced response.

III: It makes the system response independent of the input.

IV: It ensures the system response is always exponential.

V: I do not know

Question 5

Which of the following statements is true about the superposition principle in LTI systems?

Potential answers:

I: It only applies to nonlinear systems.
II: It is only valid for zero initial conditions.
III: It states that the response to a sum of inputs is the sum of the responses to each input individually.
IV: It implies that the system response is always sinusoidal.
V: I do not know

Recap of sub-module
“what is the superposition principle, and what does it imply”

- superposition principle helps logically separating specific causes into specific effects
- linear RRs $\implies$ superposition principle
- superposition principle $\implies$ "whole $=$ free $+$ forced"
- nonlinear systems WON’T satisfy this principle!

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