how to linearize a RR

Contents map

developed content units	taxonomy levels
linearization	u1, e1

prerequisite content units	taxonomy levels
RR	u1, e1

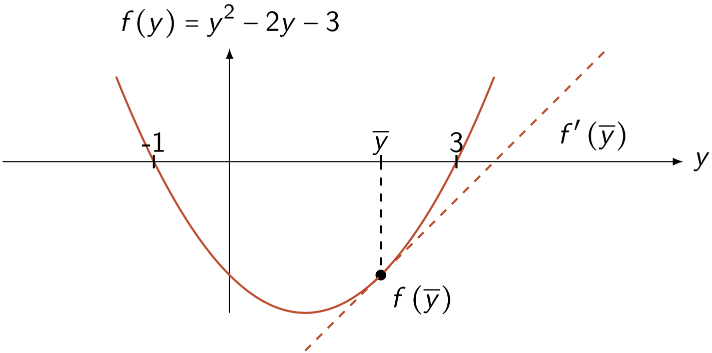
Main ILO of sub-module "how to linearize a RR"

Linearize a nonlinear RR around an equilibrium point

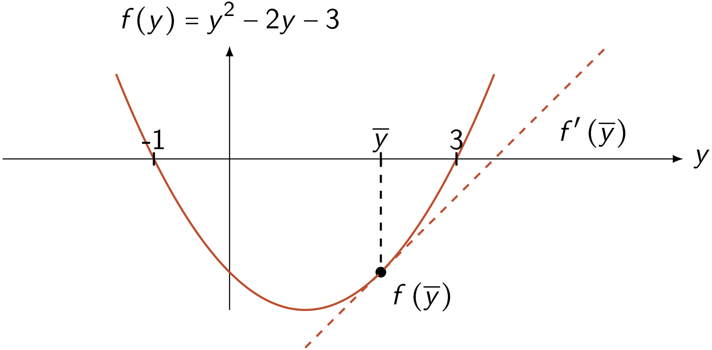
The path towards linearizing a model

- what does linearizing a function mean?
- what does linearizing a model mean?
- how shall we linearize a model?

What does linearizing a scalar function mean?



What does linearizing a scalar function mean?



$$f(y) \approx f(\overline{y}) + \left.\frac{\partial f}{\partial y}\right|_{\overline{y}} (y - \overline{y})$$

(but the approximation is valid only close to the linearization point)

Obvious requirement

(but sometimes people forget about it . . .)

to compute the approximation

$$f(y) \approx f(\overline{y}) + \left.\frac{\partial f}{\partial y}\right|_{\overline{y}} (y - \overline{y})$$

the derivative of f at $\overline{y}$ must be defined. *(notation: $f \in C^n$ means that f has all its derivatives up to order n defined in $\mathbb{R}$. $f \in C^n(\mathcal{X})$ means defined in $\mathcal{X} \subseteq \mathbb{R}$)*

What does linearizing a vectorial function mean?

$$\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^m, \quad \boldsymbol{f} \in C^1 \quad \text{enables computing} \quad \boldsymbol{f}(\boldsymbol{y}) \approx \boldsymbol{f}(\boldsymbol{y}_0) + \nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0} (\boldsymbol{y} - \boldsymbol{y}_0)$$

linearize $\Longrightarrow$ *approximate each component!*

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Discussion: then $\nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0}$ must be a matrix. Of which dimensions?

Example: linearize $\boldsymbol{f}$ around $\boldsymbol{y}_0$

$$\boldsymbol{f}\left(\boldsymbol{y}[k]\right) = \begin{bmatrix} \sin\left(y_1[k]\right) + \cos\left(y_2[k]\right) \\ \exp\left(y_1[k]y_2[k]\right) \end{bmatrix} \qquad \boldsymbol{y}_0 = \boldsymbol{y}(0) = [0, \pi]$$

And what if the vectorial function depends on more than one variable?

Assuming $\boldsymbol{f}$ differentiable in $\boldsymbol{y}_0, \boldsymbol{u}_0$,

$$\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) \approx \boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{u}_0) + \nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0} (\boldsymbol{y} - \boldsymbol{y}_0) + \nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0} (\boldsymbol{u} - \boldsymbol{u}_0)$$

with both $\nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0}$ and $\nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0}$ matrices of opportune size.

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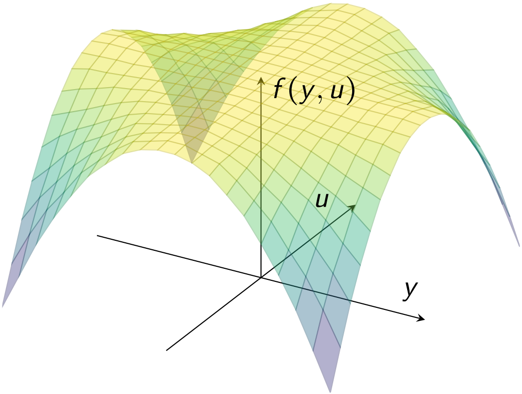
with both $\nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0}$ and $\nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{y}_0, \boldsymbol{u}_0}$ matrices of opportune size. Alternative notation:

$$\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) \approx \boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{u}_0) + \nabla \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})\Big|_{\boldsymbol{y}_0, \boldsymbol{u}_0} \begin{bmatrix} \Delta \boldsymbol{y} \\ \Delta \boldsymbol{u} \end{bmatrix}$$

Graphical example with a $\mathbb{R}^2 \mapsto \mathbb{R}$ function

$$f\left(y,u\right) \approx f\left(y_0,u_0\right) + \nabla f\left(y,u\right)\Big|_{y_0,u_0} \begin{bmatrix} \Delta x \\ \Delta u \end{bmatrix}$$



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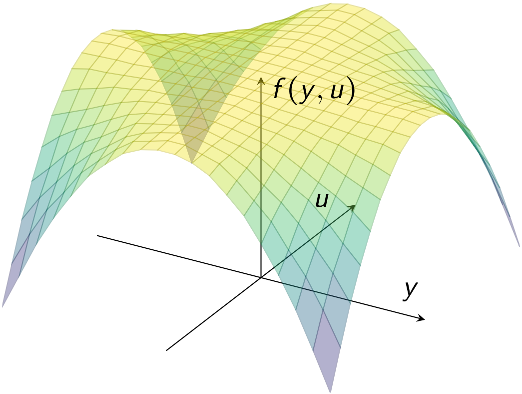

if $\boldsymbol{f} = [f_1, f_2]$ *then have two distinct plots, but the concept is the same*

Thus, linearization = stopping the Taylor series at order one

$$f \in C^M(\mathbb{R}) \qquad \Longrightarrow \qquad f(y) \approx \sum_{m=0}^{M} \frac{f^{(m)}(y_0)}{m!}(y - y_0)^m$$

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$$f \in C^M(\mathbb{R}) \qquad \Longrightarrow \qquad f(y) \approx \sum_{m=0}^{M} \frac{f^{(m)}(y_0)}{m!} (y - y_0)^m$$

multivariable extension = less neat formulas, but the concept is the same. The most

important case for our purposes:

$$\boldsymbol{f} \in C^1(\mathbb{R}^n, \mathbb{R}^m) \qquad \Longrightarrow \qquad \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) \approx \boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{u}_0) + \nabla_{\boldsymbol{y}} \boldsymbol{f}\big|_{\boldsymbol{y}_0} (\boldsymbol{y} - \boldsymbol{y}_0) + \nabla_{\boldsymbol{u}} \boldsymbol{f}\big|_{\boldsymbol{u}_0} (\boldsymbol{u} - \boldsymbol{u}_0)$$

?

What does linearizing an ODE mean?

$$\boldsymbol{y}^{+} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) \quad \approx \quad \widetilde{\boldsymbol{y}}^{+} = A\widetilde{\boldsymbol{y}} + B\widetilde{\boldsymbol{u}}$$

linearize $\Longrightarrow$ *approximate the dynamics!*

Discussion: what is the simplest way to make this linear?

$$y^+ = ay + bu^{2/3}$$

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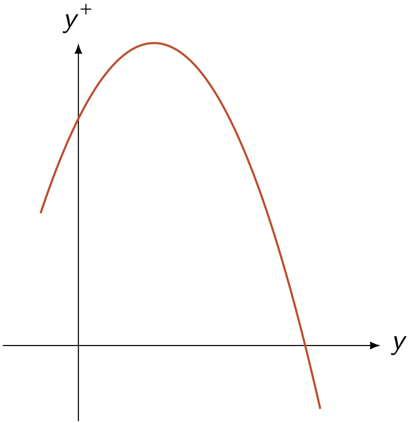
Another discussion: can we apply the same "linearization trick" to $y^+ = a\sqrt{y} + bu$?

Discussion: why do we linearize nonlinear systems?

Discussion: where do we linearize nonlinear systems? Note: main difference w.r.t. CT systems = equilibria are on the bisectors:



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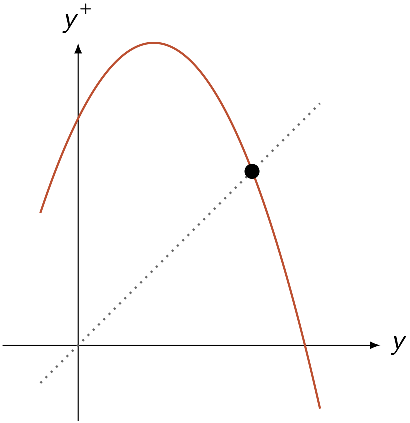

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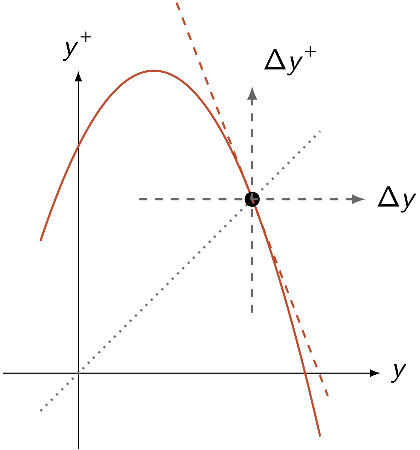

Linearization procedure - discrete time systems

$$\left(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) \text{ equilibrium} \quad \Longrightarrow \quad \boldsymbol{f}\left(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}\right) = \boldsymbol{y}_{\text{eq}}$$

Linearization procedure - discrete time systems

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Procedure (assuming that the Taylor expansion exists):

- consider $\boldsymbol{y} = \boldsymbol{y}_{\text{eq}} + \Delta\boldsymbol{y}$, and $\boldsymbol{u} = \boldsymbol{u}_{\text{eq}} + \Delta\boldsymbol{u}$
- compute

$$\boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right) \approx \boldsymbol{f}\left(\boldsymbol{y}_0, \boldsymbol{u}_0\right) + \nabla_{\boldsymbol{y}}\boldsymbol{f}\big|_{\boldsymbol{y}_0}\left(\boldsymbol{y} - \boldsymbol{y}_0\right) + \nabla_{\boldsymbol{u}}\boldsymbol{f}\big|_{\boldsymbol{u}_0}\left(\boldsymbol{u} - \boldsymbol{u}_0\right)$$

  setting though $\boldsymbol{y}_0 = \boldsymbol{y}_{\text{eq}}$

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$$\Longrightarrow \qquad \boldsymbol{y}^+ = \left(\boldsymbol{y}_{\mathrm{eq}}^+ + \Delta\boldsymbol{y}^+\right) = \boldsymbol{y}_{\mathrm{eq}} + \Delta\boldsymbol{y}^+ \approx \boldsymbol{f}\left(\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}\right) + \nabla\boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right)\Big|_{\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}} \begin{bmatrix} \Delta\boldsymbol{y} \\ \Delta\boldsymbol{u} \end{bmatrix}$$

Linearization procedure - discrete time systems

$$(\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}) \text{ equilibrium} \implies \boldsymbol{f}(\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}) = \boldsymbol{y}_{\mathrm{eq}}$$

Procedure (assuming that the Taylor expansion exists):

- consider $\boldsymbol{y} = \boldsymbol{y}_{\mathrm{eq}} + \Delta \boldsymbol{y}$, and $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{eq}} + \Delta \boldsymbol{u}$
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setting though $\boldsymbol{y}_0 = \boldsymbol{y}_{\mathrm{eq}}$

$$\implies \qquad \boldsymbol{y}^+ = \left(\boldsymbol{y}^+_{\mathrm{eq}} + \Delta \boldsymbol{y}^+\right) = \boldsymbol{y}_{\mathrm{eq}} + \Delta \boldsymbol{y}^+ \approx \boldsymbol{f}(\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}) + \nabla \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})\Big|_{\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}} \begin{bmatrix} \Delta \boldsymbol{y} \\ \Delta \boldsymbol{u} \end{bmatrix}$$

note then that $\boldsymbol{y}_{\mathrm{eq}} = \boldsymbol{f}(\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}})$, so that we can simplify the previous expression

Linearization procedure - discrete time systems

$$(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}) \text{ equilibrium} \quad \Longrightarrow$$

$$\Delta \boldsymbol{y}^{+} \approx \nabla_{\boldsymbol{y}} \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) \Big|_{\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta \boldsymbol{y} + \nabla_{\boldsymbol{u}} \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) \Big|_{\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}}} \Delta \boldsymbol{u}$$

and, since

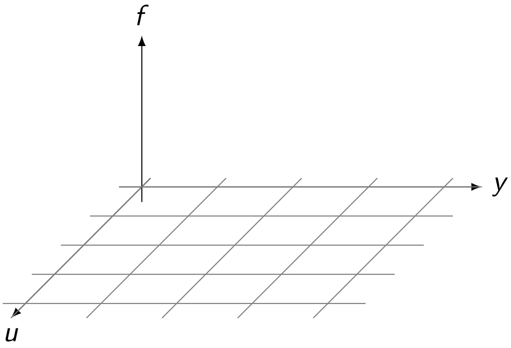
- the two ∇'s are matrices, and
- this is an approximate dynamics,

it follows that the approximated system is

$$\Delta \widetilde{\boldsymbol{y}}^{+} = A \Delta \widetilde{\boldsymbol{y}} + B \Delta \boldsymbol{u}$$

What does this mean graphically?

$$\boldsymbol{y}^{+} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right) \qquad \text{vs.} \qquad \Delta\widetilde{\boldsymbol{y}}^{+} = A\Delta\widetilde{\boldsymbol{y}} + B\Delta\widetilde{\boldsymbol{u}}$$



Summarizing the procedure

- linearizing $\boldsymbol{y}^+ = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}\right)$ is meaningful only around an equilibrium $\left(\boldsymbol{y}_{\mathrm{eq}}, \boldsymbol{u}_{\mathrm{eq}}\right)$

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- linearizing $\boldsymbol{y}^+ = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})$ is meaningful only around an equilibrium $(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$
- to find the equilibria of a system we need to solve $\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) = \boldsymbol{y}$

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- linearizing $\boldsymbol{y}^+ = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})$ is meaningful only around an equilibrium $(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$
- to find the equilibria of a system we need to solve $\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) = \boldsymbol{y}$
- each equilibrium will lead to its "own" corresponding linear model $\boldsymbol{y}^+ = A\boldsymbol{y} + B\boldsymbol{u}$, where A and B thus depend on $(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$ and $\boldsymbol{y}$, $\boldsymbol{u}$ in $\boldsymbol{y}^+ = A\boldsymbol{y} + B\boldsymbol{u}$ have actually the meaning of $\Delta\boldsymbol{y}$, $\Delta\boldsymbol{u}$ with respect to the equilibrium

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- each linearized model $\boldsymbol{y}^{+} = A\boldsymbol{y} + B\boldsymbol{u}$ is more or less valid only in a neighborhood of $(\boldsymbol{y}_{\text{eq}}, \boldsymbol{u}_{\text{eq}})$. Moreover the size of this neighborhood depends on the curvature of $\boldsymbol{f}$ around that specific equilibrium point

Recapping the rationale behind linearization

- linear systems are easier to analyze than nonlinear systems

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linearization $=$ a very useful tool to do
analysis and design of control systems

Summarizing

Linearize a nonlinear RR around an equilibrium point

- find the equilibria
- select an equilibrium
- compute the derivatives around that equilibrium
- use the formulas
- don't forget that you are also changing the coordinate system!

Most important python code for this sub-module

This will do everything for you

`https://python-control.readthedocs.io/en/latest/generated/control.linearize.html`

though it is dangerous to use tools without knowing how they work

Self-assessment material

Question 1

What is the primary purpose of linearizing a nonlinear system around an equilibrium point?

Potential answers:

I: To approximate the system's behavior in a small neighborhood of the equilibrium.
II: To completely replace the nonlinear system with a linear one globally.
III: To eliminate all nonlinearities in the system.
IV: To make the system unstable for control purposes.
V: I do not know

Question 2

Which of the following is a necessary condition for linearizing a function $f(y)$ around a point $\overline{y}$?

Potential answers:

I: The function $f(y)$ must be differentiable at $\overline{y}$.
II: The function $f(y)$ must be discontinuous at $\overline{y}$.
III: The function $f(y)$ must be constant in a neighborhood of $\overline{y}$.
IV: The function $f(y)$ must be linear globally.
V: I do not know

Question 3

What is the dimension of the Jacobian matrix $\nabla_{\boldsymbol{y}} \boldsymbol{f}$ for a vectorial function $\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^m$?

Potential answers:

I: $m \times n$

II: $n \times m$

III: $n \times n$

IV: $m \times m$

V: I do not know

Question 4

In the context of discrete-time systems, where are the equilibria located?

Potential answers:

I: On the bisector where $y = f(y)$.
II: On the bisector where $y = u$.
III: On the bisector where $f(y) = u$.
IV: On the bisector where $y = 0$.
V: I do not know

Question 5

What is the main limitation of linearizing a nonlinear system around an equilibrium point?

Potential answers:

I: The approximation is only valid in a small neighborhood of the equilibrium.
II: The linearized system is always unstable.
III: The linearized system cannot be used for control purposes.
IV: The linearized system is always globally accurate.
V: I do not know

Recap of sub-module "how to linearize a RR"

- linearization requires following a series of steps (see the summary above)
- the model that one gets in this way is an approximation of the original model
- having a graphical understanding of what means what is essential to remember how to do things
- better testing a linear controller before a nonlinear one

?