# BIBO stability for LTI systems

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# Main ILO of sub-module "BIBO stability for LTI systems"

**Graphically explain** the connection between the BIBO stability of an LTI system and its impulse response

**Find** examples of bounded inputs that map into unbounded outputs for the case of LTI systems that are not BIBO stable

# Definition: absolute integrability

$$f(t) \text{ absolutely integrable iff } \int_{-\infty}^{+\infty} |f(t)|\, dt < +\infty$$

# Question 1

Is the unit step function $\mathbb{1}(t)$ absolutely integrable over the interval $(-\infty, \infty)$?

**Potential answers:**

I: Yes, because it is bounded.

II: No, because its integral over $(-\infty, \infty)$ diverges.

III: Yes, because it is zero for $t < 0$.

IV: No, because it is discontinuous at $t = 0$.

V: I do not know.

# Question 2

Is the signal $x(t) = e^{-t}$ for $t \geq 0$, $x(t) = 0$ for $t < 0$, absolutely integrable?

**Potential answers:**

I: No, because it grows exponentially as $t \to \infty$.

II: Yes, because it is a decaying exponential.

III: Yes, because $\int_0^\infty |e^{-t}| \, dt = 1$.

IV: No, because it is not defined for $t < 0$.

V: I do not know.

# Question 3

Is the signal $x(t) = \sin(t)$ absolutely integrable over $(-\infty, \infty)$?

**Potential answers:**

I: Yes, because it is periodic.

II: No, because $\int_{-\infty}^{\infty} |\sin(t)|\, dt$ diverges.

III: Yes, because its amplitude is bounded.

IV: No, because it is not a decaying signal.

V: I do not know.

# Question 4

Is the signal $x(t) = \dfrac{1}{1+t^2}$ absolutely integrable?

**Potential answers:**

I: Yes, because $\displaystyle\int_{-\infty}^{\infty} \left|\frac{1}{1+t^2}\right| dt = \pi$.

II: No, because it is not a decaying exponential.

III: Yes, because it is symmetric about $t = 0$.

IV: No, because it is not periodic.

V: I do not know.

# Question 5

Is the signal $x(t) = te^{-t}$ for $t \geq 0$, $x(t) = 0$ for $t < 0$, absolutely integrable?

**Potential answers:**

I: No, because it grows linearly as $t \to \infty$.

II: Yes, because it is a product of a linear function and a decaying exponential.

III: Yes, because $\int_0^{\infty} |te^{-t}| \, dt = 1$.

IV: No, because it is not symmetric about $t = 0$.

V: I do not know.

# Theorem: an LTI system is BIBO stable if and only if its impulse response is absolutely integrable

BIBO stability:

$$|u(t)| < \gamma_u \quad \Longrightarrow \quad |y(t)| < \gamma_y$$

Forced response:

$$y_{\text{forced response}}(t) = h * u(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

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$$y_{\text{forced response}}(t) = h * u(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

$$\text{if } \int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < +\infty \text{ then BIBO stability}$$

# Examples: sinusoids

https://www.youtube.com/playlist?list=PL4mJLdGEHNvhCuPXsKFrnD7AaQBlMEB6a

# Examples: any non-absolutely integrable non-negative impulse response

Assumption:

$$h(t) \text{ is so that } \int_{-\infty}^{T} h(t)dt \xrightarrow{T\to+\infty} +\infty$$

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Assumption:

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implication:

$$\mathrm{step} * h(T) = y_{\mathrm{forced}}(T) = \int_{-\infty}^{T} h(t)dt \xrightarrow{T \to +\infty} +\infty :$$

# Examples: any non-absolutely integrable impulse response

Assumption:

$$h(t) \text{ is so that } \int_{-\infty}^{T} |h(t)| \, dt \xrightarrow{T \to +\infty} +\infty$$

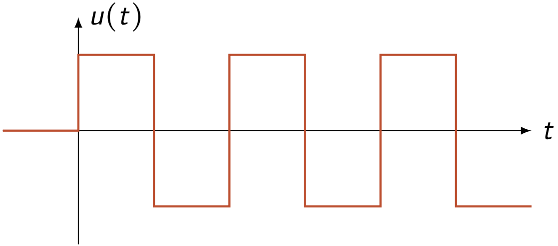
# Examples: any non-absolutely integrable impulse response

Assumption:

$$h(t) \text{ is so that } \int_{-\infty}^{T} |h(t)|\, dt \xrightarrow{T \to +\infty} +\infty$$

implication:

$$\mathrm{PWM} * h(T) = y_{\mathrm{forced}}(T) = \int_{-\infty}^{T} h(t) dt \xrightarrow{T \to +\infty} +\infty :$$



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# Summarizing

- **BIBO stable LTI system** $=$ LTI whose impulse response is absolutely integrable
- **asymptotically stable LTI system** $=$ LTI with all its equilibria asymptotically stable
- **marginally stable LTI system** $=$ LTI with all its equilibria marginally stable
- **unstable LTI system** $=$ LTI with all its equilibria unstable

all the equilibria of a given LTI system are equal in nature

# Examples: are these systems BIBO stable, marginally stable, or unstable?

- $\dot{y} = bu$
- $\dot{y} = ay + bu$ with $a < 0$
- $\dot{y} = ay + bu$ with $a > 0$

# And what about nonlinear systems?

*more complicated! You will treat this using small gain theory, in more advanced courses*

# Other related concepts

Different types of *system* stability:

- **asymptotic input-output (system) stability:** independently of $u(t)$, $x(t) \to 0$ when $t \to +\infty$
- **marginal (or simply input-output) (system) stability:** as soon as $|u(t)| < \gamma_u$, $|x(t)| < \gamma_x$ when $t \to +\infty$
- **(system) instability:** there exists at least one signal $u(t)$ for which we cannot do the bound $|x(t)| < \gamma_x$ when $t \to +\infty$

# But why do we do this?

it is necessary to know about potential instabilities, because our control system must stabilize them

# Summarizing

**Graphically explain** the connection between the BIBO stability of an LTI system and its impulse response

**Find** examples of bounded inputs that map into unbounded outputs for the case of LTI systems that are not BIBO stable

- check the absolute value of the impulse response
- if its integral is infinite, then find an input that when convolved with the original impulse response, the result gives asymptotically the absolute value version of that impulse response

Most important python code for this sub-module

# No dedicated python libraries for this ...

... but one can use the `control` library for checking the properties of the transfer function or impulse response of the system, if LTI of course

## Self-assessment material

# Question 6

Which of the following statements is true regarding the BIBO stability of an LTI system?

**Potential answers:**

I: A system is BIBO stable if its impulse response is periodic.
II: A system is BIBO stable if and only if its impulse response is absolutely integrable.
III: A system is BIBO stable if and only if all its eigenvalues have negative real parts.
IV: A system is BIBO stable if its impulse response is non-negative.
V: I do not know.

# Question 7

Which of the following impulse responses corresponds to a BIBO stable system?

**Potential answers:**

I: $h(t) = e^{t}$ for $t < 0$, $h(t) = 0$ for $t \geq 0$.

II: $h(t) = \sin(t)$.

III: $h(t) = e^{-t}\mathrm{step}(t)$, where $\mathrm{step}(t)$ is the unit step function.

IV: $h(t) = \dfrac{1}{1+t^2}$ for all $t$.

V: I do not know.

# Question 8

A system has an impulse response $h(t)$ such that $\int_{-\infty}^{+\infty} |h(t)|dt$ diverges. What does this imply?

**Potential answers:**

I: The system is asymptotically stable.
II: The system is not BIBO stable.
III: The system has a finite impulse response (FIR).
IV: The system must have at least one pole in the right-half plane.
V: I do not know.

# Question 9

Consider an LTI system with impulse response $h(t) = \dfrac{1}{1+t^2}$. What can be said about its BIBO stability?

**Potential answers:**

- I: The system is BIBO stable because its impulse response is absolutely integrable.
- II: The system is not BIBO stable because its impulse response is not causal.
- III: The system is not BIBO stable because its impulse response is not exponentially decaying.
- IV: The system is marginally stable.
- V: I do not know.

# Question 10

Which of the following statements correctly describes a BIBO unstable system?

**Potential answers:**

I: A BIBO unstable system has a stable impulse response.
II: A BIBO unstable system has a bounded output for every bounded input.
III: A BIBO unstable system has a finite impulse response.
IV: A BIBO unstable system has at least one bounded input that produces an unbounded output.
V: I do not know.

# Recap of the module "BIBO stability for LTI systems"

- for LTI systems BIBO stability is equivalent to the absolute integrability of the impulse response
- for ARMA systems BIBO stability is equivalent to having the impulse response so that all its exponential terms are vanishing in time
- for nonlinear systems one shall use more advanced tools that will be seen in later on courses

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