explain and determine the convergence properties of an equilibrium

Contents map

developed content units	**taxonomy levels**
convergent equilibrium	u1, e1
asymptotically stable equilibrium	u1, e1

prerequisite content units	**taxonomy levels**
ODE	u1, e1
equilibrium	u1, e1
marginally stable equilibrium	u1, e1
simply stable equilibrium	u1, e1

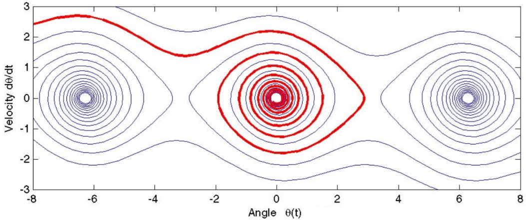
Main ILO of sub-module
“explain and determine the convergence properties of an equilibrium”

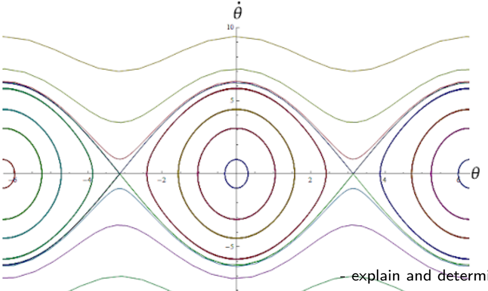
Graphically explain the definition of convergent equilibrium and interpret its practical meaning

Give examples of systems that have convergent equilibria or not, and discuss scenarios where convergence is important

Determine if an equilibrium is convergent or not from inspecting a phase portrait, based on the flow near equilibrium

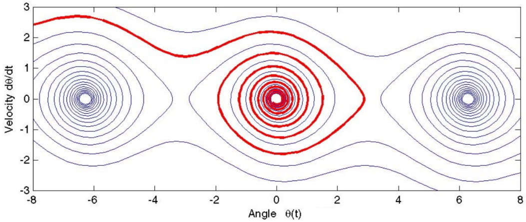
Intuition: if I perturb a little bit this system from its equilibrium, where will it eventually end up?





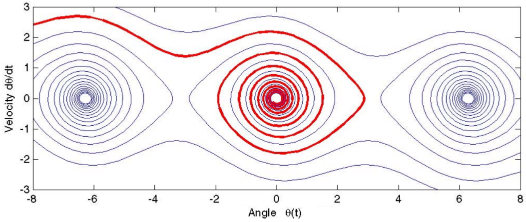
main point of convergence of an equilibrium: where does the system end up, eventually, if starting closeby?

Convergent equilibrium (continuous time case, formally)



$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$ $\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}_e\right)$ $\left(\boldsymbol{y}_e, \boldsymbol{u}_e\right) =$ equilibrium

Convergent equilibrium (continuous time case, formally)

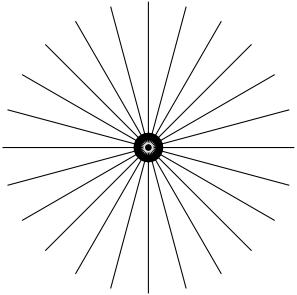


$\overline{\boldsymbol{u}} = \boldsymbol{u}_e = const.$ $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}_e)$ $(\boldsymbol{y}_e, \boldsymbol{u}_e) =$ equilibrium

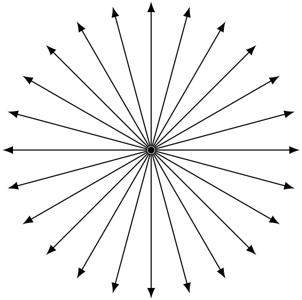
Definition (convergent equilibrium)

$\boldsymbol{y}_e$ = *is convergent if there is a neighborhood containing that equilibrium so that if one starts from within that neigborhood, eventually* $\boldsymbol{y}(t) \xrightarrow{t \to +\infty} \boldsymbol{y}_e$

Some phase portraits to exemplify the potential situations



Some phase portraits to exemplify the potential situations



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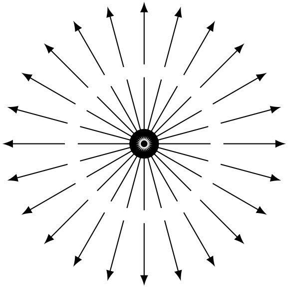

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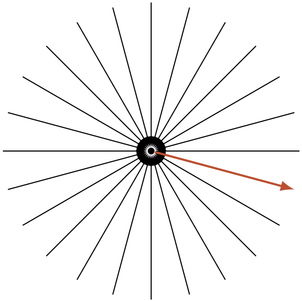
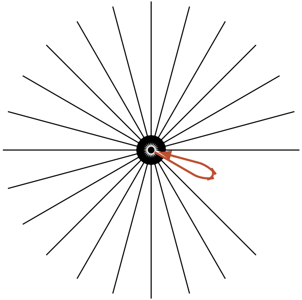
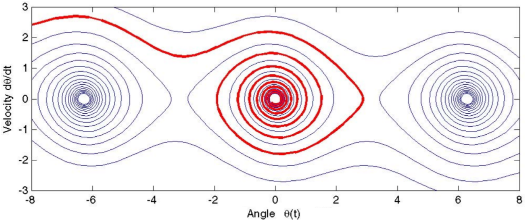


Some phase portraits to exemplify the potential situations



Important differences



- **simply stable equilibrium:** we can confine arbitrarily the trajectories around the equilibrium by reducing ε opportunely
- **convergent equilibrium:** we do not care about whether we can confine arbitrarily the trajectories or not; it may or not happen, it does not matter. We though know that if we start close enough then *eventually* the distance $\|\boldsymbol{y}(t) - \boldsymbol{y}_e\|$ will go to zero

Discussion: Consider the continuous time system

$$\dot{y}(t) = \begin{cases} +y(t) & \text{if } t < 10^5 \\ -y(t) & \text{otherwise.} \end{cases}$$

Which type of equilibrium is 0? Possibilities:

- just simply stable
- just convergent
- both
- none
- I do not know

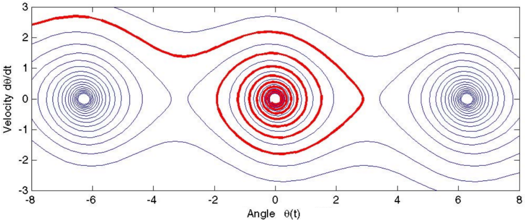
Discussion: Consider the continuous time system

$$\dot{y}(t) = \begin{cases} +y(t) & \text{if } \sup_{T \in (-\infty, t)} |y(T)| < 1 \\ -y(t) & \text{otherwise.} \end{cases}$$

Which type of equilibrium is 0? Possibilities:

- just simply stable
- just convergent
- both
- none
- I do not know

Asymptotic stability

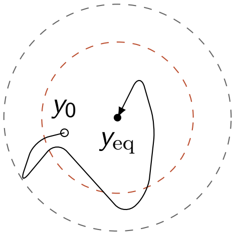


$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \overline{\boldsymbol{u}}\right) \qquad \boldsymbol{y}_e = \text{equilibrium}$$

Definition (asymptotically stable equilibrium)

the equilibrium $\boldsymbol{y}_e$ is said to be asymptotically stable if it is simultaneously simply stable & convergent

Asymptotic stability in practice



Definition (asymptotically stable equilibrium)

the equilibrium $\boldsymbol{y}_e$ *is said to be asymptotically stable if it is simultaneously simply stable & convergent*

Discussion: how is the origin for a spring-mass system with friction? And the downwards-equilibrium for a pendulum with friction?

Summarizing

Graphically explain the definition of convergent equilibrium and interpret its practical meaning

Give examples of systems that have convergent equilibria or not, and discuss scenarios where convergence is important

Determine if an equilibrium is convergent or not from inspecting a phase portrait, based on the flow near equilibrium

- convergence deals with what happens at the end of time, while marginal stability deals with what happens in the transient
- an equilibrium is eventually convergent if it has a so-called 'basin of attraction', i.e., a zone of initial conditions in the phase portrait for which starting in that zone makes the free evolution end up in that equilibrium, eventually

Most important python code for this sub-module

No dedicated python toolbox for tihs

. . . but `scipy.linalg` can be used to analyse stability for LTI systems

Self-assessment material

Question 1

If an equilibrium is not convergent, then is it necessarily unstable?

Potential answers:

I: Yes, non-convergent equilibria imply instability.
II: No, an equilibrium can be non-convergent but still stable.
III: No, all equilibria are inherently convergent.
IV: It depends on whether the system is linear.
V: I do not know

Question 2

If an equilibrium is convergent, does that mean it is marginally stable?

Potential answers:

I: Yes, convergence always implies marginal stability.
II: No, convergence does not necessarily mean marginal stability.
III: Only if the system has no damping.
IV: Only in linear time-invariant systems.
V: I do not know

Question 3

If an equilibrium is both convergent and marginally stable, then is it asymptotically stable?

Potential answers:

I: Yes, marginal stability plus convergence implies asymptotic stability.
II: No, asymptotic stability requires trajectories to decay to equilibrium.
III: Yes, unless external forces are applied.
IV: Only in conservative systems.
V: I do not know

Question 4

If an equilibrium is marginally stable, then is it necessarily convergent?

Potential answers:

I: Yes, marginal stability implies convergence.
II: No, marginal stability does not guarantee convergence.
III: Only in discrete-time systems.
IV: Only for systems with no external perturbations.
V: I do not know

Question 5

Is the origin for the Lotka-Volterra model convergent?

Potential answers:

I: No, it is a saddle point and therefore unstable.
II: Yes, because populations always return to equilibrium.
III: Yes, because it has only non-positive eigenvalues.
IV: It depends on the initial conditions.
V: I do not know

Question 6

Is the non-null equilibrium for the Lotka-Volterra model asymptotically stable?

Potential answers:

I: Yes, the non-null equilibrium always attracts trajectories.
II: No, it is typically neutrally stable with closed orbits.
III: No, it is always unstable.
IV: It depends on the values of the system parameters.
V: I do not know

Recap of sub-module “explain and determine the convergence properties of an equilibrium”

- convergence is disconnected from “marginal stability”, since in general one may have one case and not the other, and viceversa, or both, or none
- the concept of convergence focuses on the limit behavior, ignoring the transient

?