Systems Laboratory, Spring 2025

Damiano Varagnolo –

state space representations

Contents map

developed content units	taxonomy levels
state of a system	u1, e1
separation principle	u1, e1

prerequisite content units	taxonomy levels
ODE	u1, e1

Main ILO of sub-module "state space representations"

Define the meaning of "state space representation" in the context of linear and non-linear dynamical systems

Discussion: which information do you need to forecast accurately how long you may use your cellphone before its battery hits 0%?

Summarizing

these pieces of information contain all I need to forecast the future evolution of the battery level:

- current level of charge of the battery
- how much I will use the phone in the future
- how healthy the battery of my phone is
- which environmental factors may induce additional effects (too warm, too cold)

A simple model of the battery charge as a dynamical system

$$\text{Time Remaining} = \frac{\text{Discharge Rate}}{\text{Remaining Capacity}}$$

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rewriting as an ODE:

- $y(t) = Q(t) =$ remaining battery capacity at time t (mAh)
- $u(t) = I(t) =$ current discharge rate at time t (mA)

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$$\implies \quad \dot{y} = -u$$

What is a state?

$$\begin{cases} \dot{x} &= -u \\ y = x \end{cases}$$

"the current value of the state $x(t)$ contains all the information necessary to forecast the future evolution of the output $y(t)$ and of the state $x(t)$, assuming to know the future $u(t)$. I.e., to compute the future values $y(t+\tau)$ and $x(t+\tau)$ it is enough to know the current $x(t)$ and the current and future inputs $u(t : t+\tau)$"

Question 1

In a spring-mass system, which of the following is a valid state variable?

Potential answers:

I: The temperature of the spring.
II: The displacement of the mass from its equilibrium position.
III: The color of the mass.
IV: The external force applied to the system.
V: I do not know.

Question 2

Which of the following pairs of variables can fully describe the state of a spring-mass system?

Potential answers:

I: The mass of the spring and the stiffness of the mass.
II: The external force and the displacement of the mass.
III: The displacement of the mass and the velocity of the mass.
IV: The acceleration of the mass and the color of the spring.
V: I do not know.

What do we mean with “modelling a state-space dynamical system”?

Defining

$$\begin{cases} \dot{\boldsymbol{x}} & = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\theta}\right) \\ \boldsymbol{y} & = \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\theta}\right) \end{cases}$$

and

- the variables
  - $\boldsymbol{u}$ = the inputs
  - $\boldsymbol{d}$ = the disturbances
  - $\boldsymbol{x}$ = the states vector
  - $\boldsymbol{y}$ = the measured outputs
- the structure of the functions $\boldsymbol{f}$ and $\boldsymbol{g}$
- the value of the parameters $\boldsymbol{\theta}$

State space model - definition

Ingredients:

- the number of inputs, outputs and state variables must be finite
- the differential equations must be first order
- the separation principle *(the current value of the state contains all the information necessary to forecast the future evolution of the outputs and of the state)* shall be satisfied

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state space model = finite set of first-order differential equations that connect a finite set of inputs, outputs and state variables so that they satisfy the separation principle

State space representations - Notation

- $u_1, \ldots, u_m$ = inputs
- $x_1, \ldots, x_n$ = states
- $y_1, \ldots, y_p$ = outputs
- $d_1, \ldots, d_q$ = disturbances

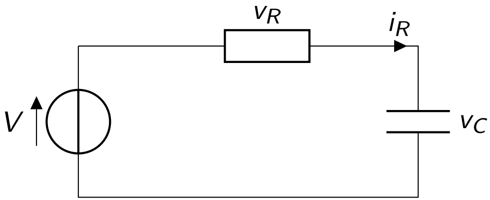
State space representations - Notation

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\
\boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})
\end{aligned}
$$

- $\boldsymbol{f}$ = state transition map
- $\boldsymbol{g}$ = output map

examples

RC-circuit



$$\dot{v}_C = -\frac{1}{RC} v_C + \frac{1}{RC} V \qquad (1)$$

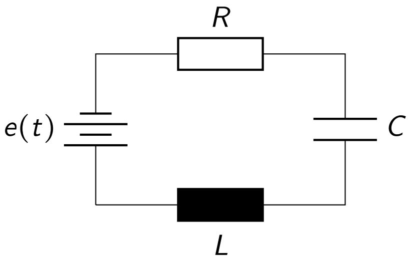
or, using control-oriented names,

$$\dot{x} = -\frac{1}{RC} x + \frac{1}{RC} u \qquad y = x \qquad (2)$$

Generalization: exponential growth, scalar version

$$\dot{x} = \alpha x + \beta u \qquad y = x \tag{3}$$

Generalizing in an other way: RCL-circuits



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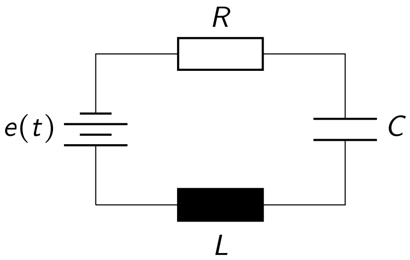

EOM: Kirchhoff laws $\Longrightarrow$ $v_L(t) = L\frac{di(t)}{dt} \quad v_i(t) = Ri(t) \quad v_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$

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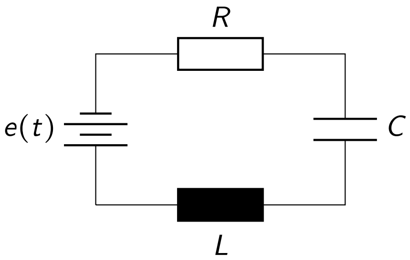

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$$e(t) = v_L(t) + v_R(t) + v_C(t) \implies e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau \quad (4)$$

Generalizing in an other way: RCL-circuits part two

$$e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau \tag{5}$$

can be rewritten as

$$\begin{cases} \dot{\left(\int_0^t i(\tau)d\tau\right)} & = i(t) \\ \dot{i}(t) & = \frac{1}{L}e(t) - \frac{R}{L}i(t) - \frac{1}{LC}\int_0^t i(\tau)d\tau \end{cases} \tag{6}$$

Generalizing in an other way: RCL-circuits part two

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that can be rewritten as

$$\begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = \frac{1}{L}u(t) - \frac{R}{L}x_2 - \frac{1}{LC}x_1 \end{cases} \qquad y = x_2 \qquad (7)$$

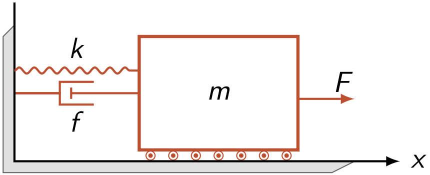
Exponential growth, matricial version

Generalization of all linear systems, thus also of "RCL circuits"

$$\begin{cases} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} \end{cases}$$

Spring-mass systems

E.g., position of a cart fastened with a spring to a wall and subject to friction



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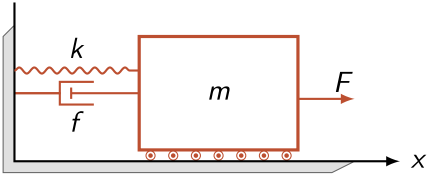

EOM:

- force from the spring: $F_x(t) = -kx(t)$
- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

Spring-mass systems

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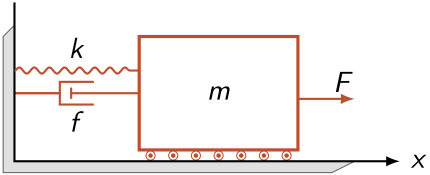

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  $m\ddot{x}(t) = F_x(t) + F_f(t) + F(t)$

Spring-mass systems

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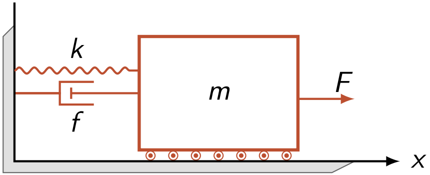

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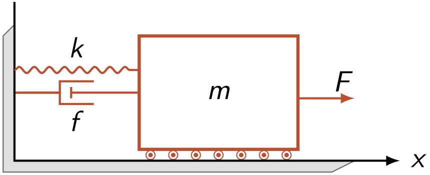
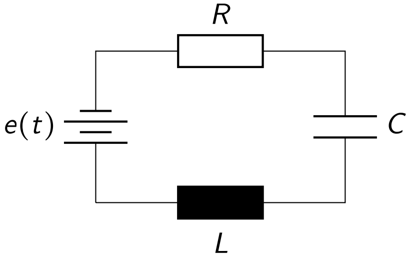
- force from the spring: $F_x(t) = -kx(t)$
- friction: $F_f(t) = -f\dot{x}(t)$
- applied force: $F(t)$
- Newton's second law: $\sum F = m\ddot{x}(t)$

$$m\ddot{x}(t) = F_x(t) + F_f(t) + F(t) \quad \mapsto \quad m\ddot{x}(t) = -kx(t) - f\dot{x}(t) + F(t)$$

(rewritable again as $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$)

Important message: these two systems are "the same"





thus studying $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ in means studying both systems simultaneously!

Lotka-Volterra

- $y_{\text{prey}} :=$ prey
- $y_{\text{pred}} :=$ predator

$$\begin{cases} \dot{y}_{\text{prey}} &= \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} &= -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

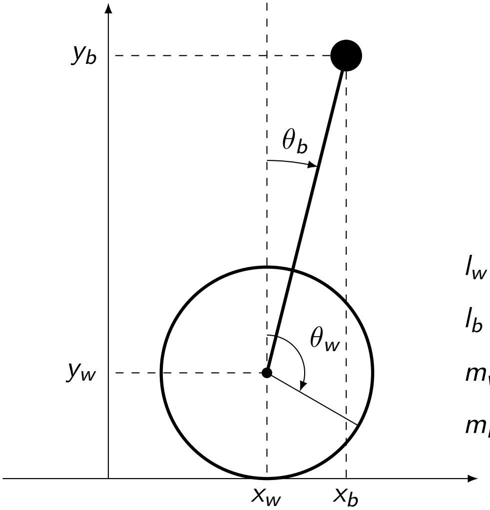
`./LotkaVolterraSimulator.ipynb`

Van-der-Pol oscillator

$$\begin{cases} \dot{x}_1 & = \mu \left(x_1 - \frac{x_1^3}{3} - x_2 \right) \\ \dot{x}_2 & = \frac{x_1}{\mu} \end{cases} \tag{8}$$

`./VanDerPolSimulator.ipynb`

Balancing robot



l_w = radius of the wheel

l_b = body-wheel's center distance

m_w = mass of the wheel

m_b = mass of the body

Balancing robot

$$\begin{aligned}
\left(I_b + m_b l_b^2\right)\ddot{\theta}_b &= +m_b l_b g \sin\left(\theta_b\right) - m_b l_b \ddot{x}_w \cos\left(\theta_b\right) - \frac{K_t}{R_m} v_m + \left(\frac{K_e K_t}{R_m} + b_f\right)\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \\
\left(\frac{I_w}{l_w} + l_w m_b + l_w m_w\right)\ddot{x}_w &= -m_b l_b l_w \ddot{\theta}_b \cos\left(\theta_b\right) + m_b l_b l_w \dot{\theta}_b^2 \sin\left(\theta_b\right) + \frac{K_t}{R_m} v_m - \left(\frac{K_e K_t}{R_m} + b_f\right)\left(\frac{\dot{x}_w}{l_w} - \dot{\theta}_b\right)
\end{aligned} \quad (9)$$

Insulin concentration

- $x_1 :=$ sugar concentration
- $x_2 :=$ insulin concentration
- $u_1 :=$ food intake
- $u_2 :=$ insulin intake
- $c :=$ sugar concentration in fasting *(person-specific)*

$$\begin{cases} \dot{x}_2 = a_{21}\left(x_1 - c\right) - a_{22}x_2 + b_2 u_2 & x_1 \geq c \\ \dot{x}_2 = -a_{22}x_2 + b_2 u_2 & x_1 < c \end{cases}$$

$$\begin{cases} \dot{x}_1 = -a_{11}x_1x_2 - a_{12}\left(x_1 - c\right) + b_1 u_1 & x_1 \geq c \\ \dot{x}_1 = -a_{11}x_1x_2 + b_1 u_1 & x_1 < c \end{cases}$$

Summarizing

Define the meaning of "state space representation" in the context of linear and non-linear dynamical systems

- recall the definition of state space model
- be sure to have interiorized the separation principle with some practical examples

Most important python code for this sub-module

Important library

`https://python-control.readthedocs.io/en/0.10.1/conventions.html#state-space-systems`

Self-assessment material

Question 3

What is the primary purpose of the separation principle in state space representations?

Potential answers:

I: To ensure that the system has an infinite number of states.
II: To eliminate the need for inputs in the system model.
III: To ensure that the current state contains all information needed to predict future behavior.
IV: To simplify the computation of system eigenvalues.
V: I do not know.

Question 4

Which of the following is a valid state variable in a state space representation of a dynamical system?

Potential answers:

I: The external force applied to the system.
II: The displacement of a mass in a spring-mass system.
III: The color of the system components.
IV: The temperature of the environment.
V: I do not know.

Question 5

What does the state transition map $\boldsymbol{f}$ in a state space representation describe?

Potential answers:

I: The relationship between inputs and outputs.
II: The evolution of the state variables over time.
III: The effect of disturbances on the system.
IV: The stability of the system.
V: I do not know.

Question 6

What is the role of the output map $\boldsymbol{g}$ in a state space representation?

Potential answers:

I: To define the system's stability.
II: To describe the evolution of the state variables.
III: To relate the state variables and inputs to the measured outputs.
IV: To eliminate the need for disturbances in the model.
V: I do not know.

Question 7

Which of the following pairs of variables is sufficient to describe the state of a simple pendulum system?

Potential answers:

I: The mass of the pendulum and the length of the string.
II: The external torque and the angular displacement.
III: The angular displacement and the angular velocity.
IV: The color of the pendulum and the gravitational constant.
V: I do not know.

Exercise: find which parts of these paragraphs are correct and which ones are wrong

The RCL circuit can be modeled by a second-order linear differential equation where the inductance, resistance, and capacitance determine the system's resonance frequency. Interestingly, in an underdamped RCL circuit, the system will always return to equilibrium without oscillating, which reflects the energy dissipation in the resistor.

Exercise: find which parts of these paragraphs are correct and which ones are wrong

The Lotka-Volterra model is a non-linear system that describes interactions between two species: one as a predator and the other as prey. The model assumes that the growth rate of the prey population is proportional to the current population size, which would mean that the population would grow indefinitely in the absence of predators. Similarly, the predator population is dependent solely on the availability of prey, implying that predators could not survive without prey even if there were other food sources available.

Exercise: find which parts of these paragraphs are correct and which ones are wrong

The Van der Pol oscillator is an example of a non-linear system that exhibits limit cycle behavior. This behavior is critical as it shows how the system can maintain a stable oscillation regardless of initial conditions, which is a feature not present in linear oscillators. It's important to note that the Van der Pol oscillator can only have a single limit cycle, and any perturbations will lead to a quick return to this cycle, indicating that the system is highly stable.

Recap of sub-module "state space representations"

- a set of variables is a state vector if it satisfies for that model the separation principle, i.e., the current state vector "decouples" the past with the future
- state space models are finite, and first order vectorial models

?