Systems Laboratory, Spring 2025

Damiano Varagnolo –

computing free evolutions and forced responses of LTI systems

Contents map

developed content units	taxonomy levels
free evolution	u1, e1
forced response	u1, e1

prerequisite content units	taxonomy levels
LTI ODE	u1, e1
convolution	u1, e1
partial fraction decomposition	u1, e1

Main ILO of sub-module "computing free evolutions and forced responses of LTI systems"

Compute free evolutions and forced responses of LTI systems using Laplace-based formulas (but only as procedural tools)

Disclaimer

the formulas introduced in this module shall be taken as "ex machina"

Focus in this module = on ARMA models

$$y^{(n)} = a_{n-1} y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots + b_0 u$$

with $^{(i)}$ meaning the i-th time derivative.

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Laplace transforms - links for who would like to get more info about them

Laplace transforms = extension of Fourier transforms; interesting material:

- `https://www.youtube.com/watch?v=r6sGWTCMz2k` (Fourier series)
- `https://www.youtube.com/watch?v=spUNpyF58BY` (Fourier transforms)
- `https://www.youtube.com/watch?v=nmgFG7PUHfo` (on the historical importance of Fast Fourier Transforms)
- `https://www.youtube.com/watch?v=7UvtU75NXTg` (Laplace Transforms, in math)
- `https://www.youtube.com/watch?v=n2y7n6jw5d0` (Laplace Transforms, graphically)

Main usefulness: convolution in time transforms into multiplication in Laplace-domain, and viceversa

$$\begin{cases} H(s) = \mathcal{L}\left\{h(t)\right\} \\ U(s) = \mathcal{L}\left\{u(t)\right\} \end{cases} \implies \mathcal{L}\left\{h * u(t)\right\} = H(s)U(s)$$

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Noticeable name: *transfer function* $(= H(s) = \mathcal{L}\{\text{impulse response}\})$

An intuitive explanation of the usefulness of the Laplace transform in automatic control

initial value problem

$$\dot{x}(t) + x(t) = 0, \quad x(0) = 2$$

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algebraic problem

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$$sX(s) - x(0) + X(s) = 0$$

algebra

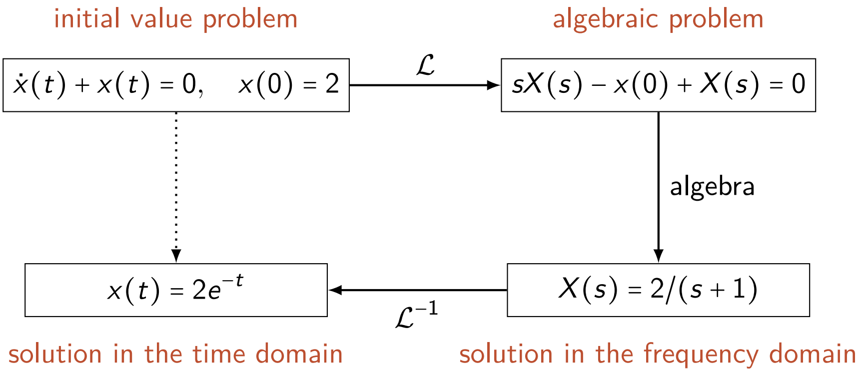
$$X(s) = 2/(s+1)$$

solution in the frequency domain

$$x(t) = 2e^{-t}$$

solution in the time domain

An intuitive explanation of the usefulness of the Laplace transform in automatic control



First set of formulas to memorize: Laplace-transforming derivatives

(these will be motivated in other courses)

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$$\mathcal{L}\left\{x^m\right\} = \ldots$$

Example: spring mass system

$$\ddot{y} = -\frac{f}{m}\dot{y} - \frac{k}{m}y + u$$

$$\Downarrow$$

$$s^2 Y(s) - sy_0 - \dot{y}_0 = -\frac{f}{m}\Big(sY(s) - y_0\Big) - \frac{k}{m}Y(s) + U(s)$$

$$\Downarrow$$

$$s^2 Y(s) + \frac{f}{m}sY(s) + \frac{k}{m}Y(s) = +sy_0 + \dot{y}_0 + \frac{f}{m}y_0 + U(s)$$

$$\Downarrow$$

$$Y(s) = \frac{y_0\left(\frac{f}{m} + s\right) + \dot{y}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} + \frac{1}{s^2 + \frac{f}{m}s + \frac{k}{m}}U(s)$$

And what shall we do once we get this?

$$\text{generalizing the previous slide: } Y(s) = \frac{M(s)}{A(s)} + \frac{B(s)}{A(s)}U(s)$$

with

- $\dfrac{M(s)}{A(s)}$ = Laplace transform of the free evolution
- $\dfrac{B(s)}{A(s)}U(s)$ = Laplace transform of the forced response

$\Longrightarrow$ we shall anti-transform; how?

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$\Longrightarrow$ we shall anti-transform; how? Main 2 cases:

- either $U(s) = \frac{\text{polynomial in } s}{\text{polynomial in } s}$
- or $U(s)$ = something else

Question 1

Is the Laplace transform of the signal

$$h(t) = \begin{cases} \dfrac{1}{t+1} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

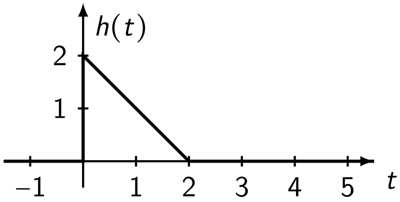
a rational Laplace transform?

Potential answers:

I: yes
II: no
III: it depends
IV: I don't know

Question 2

Is the Laplace transform of the signal $h(t)$ below a rational Laplace transform?



Potential answers:

I: yes

II: no

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IV: I don't know

first case: rational $U(s)$

How to do if $U(s) = \dfrac{\text{polynomial in } s}{\text{polynomial in } s}$

$$Y(s) = \frac{M(s)}{A(s)} + \frac{B(s)}{A(s)} U(s) \quad \mapsto \quad Y(s) = \frac{M(s)}{A(s)} + \frac{C(s)}{D(s)}$$

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write each of the two parts of the signal as

$$\frac{N(s)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots}$$

Next step: partial fraction decomposition

- **case single poles:** if $\dfrac{N(s)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots}$ is s.t. $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots$ then there exist $\alpha_1, \alpha_2, \alpha_3, \ldots$ s.t.

$$\frac{N(s)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots} = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2} + \frac{\alpha_3}{s-\lambda_3} + \cdots \tag{1}$$

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- **case repeated poles:** if some poles are repeated, then there exist $\alpha_{1,1}, \ldots, \alpha_{1,n1}, \alpha_{2,1}, \ldots, \alpha_{2,n2}, \ldots,$ s.t.

$$\frac{N(s)}{(s-\lambda_1)^{n1}(s-\lambda_2)^{n2}\ldots} = \frac{\alpha_{1,1}}{s-\lambda_1} + \ldots + \frac{\alpha_{1,n1}}{(s-\lambda_1)^{n1}} + \frac{\alpha_{2,1}}{s-\lambda_2} + \ldots + \frac{\alpha_{2,n2}}{(s-\lambda_2)^{n2}} + \ldots \tag{2}$$

Next step: partial fraction decomposition

- **case single poles:** if $\frac{N(s)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots}$ is s.t. $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots$ then there exist $\alpha_1, \alpha_2, \alpha_3, \ldots$ s.t.

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$$\frac{N(s)}{(s-\lambda_1)^{n1}(s-\lambda_2)^{n2}\ldots} = \frac{\alpha_{1,1}}{s-\lambda_1} + \ldots + \frac{\alpha_{1,n1}}{(s-\lambda_1)^{n1}} + \frac{\alpha_{2,1}}{s-\lambda_2} + \ldots + \frac{\alpha_{2,n2}}{(s-\lambda_2)^{n2}} + \ldots \tag{2}$$

"But how do I compute α_1, α_2, etc.?" $\mapsto$

`en.wikipedia.org/wiki/Partial_fraction_decomposition`

(tip: start from `en.wikipedia.org/wiki/Heaviside_cover-up_method`)

Anti-transforming in the rational $U(s)$ case

if $Y(s) = \frac{\alpha_{1,1}}{s - \lambda_1} + \ldots + \frac{\alpha_{1,n1}}{(s - \lambda_1)^{n1}} + \frac{\alpha_{2,1}}{s - \lambda_2} + \ldots + \frac{\alpha_{2,n2}}{(s - \lambda_2)^{n2}} + \ldots$ then use

$$\mathcal{L}\left\{t^n e^{\lambda t}\right\} = \frac{n!}{(s - \lambda)^{n+1}} \quad \leftrightarrow \quad \mathcal{L}^{-1}\left\{\frac{n!}{(s - \lambda)^{n+1}}\right\} = t^n e^{\lambda t}$$

Numerical Example: Inverse Laplace Transform of a Rational Function

$$Y(s) = \frac{3}{s-2} + \frac{4}{(s-2)^2} + \frac{5}{s+1}$$

goal = compute the inverse Laplace transform $y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\}$

Step 1: Identify the terms

$$Y(s) = \frac{3}{s-2} + \frac{4}{(s-2)^2} + \frac{5}{s+1}$$

Here:

- $\lambda_1 = 2$, with coefficients $\alpha_{1,1} = 3$ and $\alpha_{1,2} = 4$
- $\lambda_2 = -1$, with coefficient $\alpha_{2,1} = 5$

Step 2: Apply the inverse Laplace transform formula

by means of

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s-\lambda)^{n+1}}\right\} = t^n e^{\lambda t}$$

we compute the inverse Laplace transform of each term:

- $\mathcal{L}^{-1}\left\{\frac{3}{s-2}\right\} = 3e^{2t}$
- $\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2}\right\} = 4te^{2t}$
- $\mathcal{L}^{-1}\left\{\frac{5}{s+1}\right\} = 5e^{-t}$

Step 3: Combine the results

then we have that the inverse Laplace transform $y(t)$ is the sum of the individual transforms, i.e.,

$$y(t) = 3e^{2t} + 4te^{2t} + 5e^{-t}$$

Another Example: Inverse Laplace Transform with Complex Conjugate Terms

let

$$Y(s) = \frac{2s + 3}{s^2 + 2s + 5}$$

and the goal to be to compute the inverse Laplace transform $y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\}$

Step 1: Factor the denominator

note: $s^2 + 2s + 5$ has complex conjugate roots, indeed

$$s^2 + 2s + 5 = (s+1)^2 + 4$$

and thus

$$Y(s) = \frac{2s+3}{(s+1)^2 + 4}$$

Step 2: Express in terms of standard forms

rewrite $Y(s)$ to match the standard forms for inverse Laplace transforms involving complex conjugates, i.e.,

$$Y(s) = \frac{2(s+1)+1}{(s+1)^2+4} = 2 \cdot \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}.$$

Step 3: Apply the inverse Laplace transform formula

since

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos(bt),$$

$$\mathcal{L}^{-1}\left\{\frac{b}{(s+a)^2+b^2}\right\} = e^{-at}\sin(bt),$$

we have, for the various terms:

- $\mathcal{L}^{-1}\left\{2 \cdot \frac{s+1}{(s+1)^2+4}\right\} = 2e^{-t}\cos(2t)$
- $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = \frac{1}{2}e^{-t}\sin(2t)$

Step 4: Combine the results

$$y(t) = 2e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)$$

Extremely important result

a LTI in free evolution behaves as a combination of terms $e^{\lambda t}$, $te^{\lambda t}$, $t^2 e^{\lambda t}$, etc. for a set of different λ's and powers of t, called the *modes* of the system

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Discussion: assuming that we have two modes, $e^{-0.3t}$ and $e^{-1.6t}$, so that

$$y(t) = \alpha_1 e^{-0.3t} + \alpha_2 e^{-1.6t}.$$

What determines α_1 and α_2?

second case: irrational $U(s)$

In this case we cannot use partial fractions decompositions as before

from $Y(s) = \dfrac{M(s)}{A(s)} + \dfrac{B(s)}{A(s)} U(s)$ we follow the algorithm

- find $y_{\mathrm{free}}(t)$ from PFDs of $\dfrac{M(s)}{A(s)}$ as before
- find the impulse response $h(t)$ from PFDs of $\dfrac{B(s)}{A(s)}$ as before
- find $y_{\mathrm{forced}}(t)$ as $h * u(t)$

Summarizing

Compute free evolutions and forced responses of LTI systems using Laplace-based formulas (but only as procedural tools)

- Laplace the ARMA
- if $u(t)$ admits a rational $U(s)$ then write $Y(s) = \frac{\text{polynomial}}{\text{polynomial}}$, do PFD, and do inverse-Laplaces
- if $u(t)$ does not admit a rational $U(s)$, do similarly as before but do PFD only for the free evolution and impulse response, and find the forced response by means of convolution

Most important python code for this sub-module

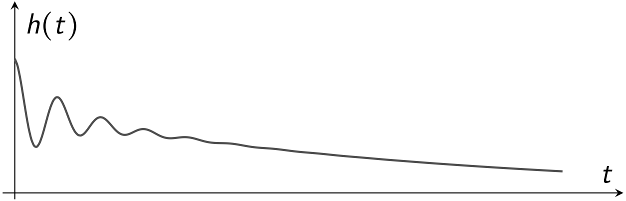
Two essential libraries

- https://python-control.readthedocs.io/en/0.10.1/generated/control.modal_form.html
- https://docs.sympy.org/latest/modules/physics/control/lti.html

Self-assessment material

Question 3

Which type of LTI system may produce the impulse response $h(t)$ represented in the picture?



Potential answers:

I: first order

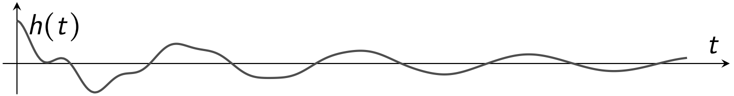
II: second order

III: at least third order

IV: I do not know

Question 4

Which type of LTI system may produce the impulse response $h(t)$ represented in the picture?

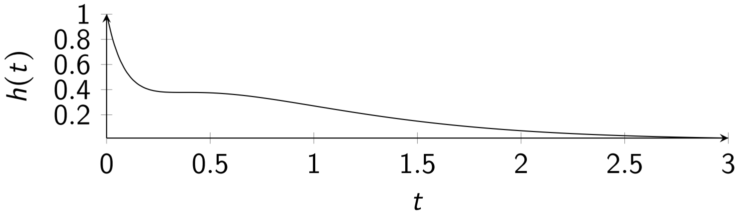


Potential answers:

I: first order
II: second order
III: third order
IV: at least fourth order
V: I do not know

Question 5

Which type of LTI system may produce the impulse response $h(t)$ below?



Potential answers:

I: first order

II: second order

III: at least third order

IV: I do not know

Question 6

What is the primary purpose of using Laplace transforms in solving LTI systems?

Potential answers:

I: To convert differential equations into algebraic equations for easier solving.
II: To transform convolution in the time domain into multiplication in the Laplace domain.
III: To directly compute the eigenvalues of the system matrix.
IV: To eliminate the need for initial conditions in solving differential equations.
V: I do not know.

Question 7

What is the correct form of the inverse Laplace transform of $\dfrac{1}{(s-\lambda)^2}$?

Potential answers:

I: $e^{\lambda t}$
II: $te^{\lambda t}$
III: $te^{\lambda t}$
IV: $\frac{1}{2}t^2e^{\lambda t}$
V: I do not know.

Question 8

What is the inverse Laplace transform of $\dfrac{s+1}{(s+1)^2+4}$?

Potential answers:

I: $e^{-t}\sin(2t)$
II: $e^{-t}\cos(2t)$
III: $e^{-t}\cos(t)$
IV: $e^{-t}\sin(t)$
V: I do not know.

Question 9

In the ARMA model $y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots + b_0 u$, why is the leading coefficient of $y^{(n)}$ typically set to 1?

Potential answers:

I: To ensure the system is stable.
II: To simplify the computation of eigenvalues.
III: To reduce the number of parameters and work with monic polynomials.
IV: To make the system linear time-invariant.
V: I do not know.

Question 10

What determines the coefficients α_1 and α_2 in the free evolution response $y(t) = \alpha_1 e^{-0.3t} + \alpha_2 e^{-1.6t}$?

Potential answers:

I: The eigenvalues of the system matrix.
II: The input signal $u(t)$.
III: The initial conditions of the system.
IV: The poles of the transfer function.
V: I do not know.

Recap of sub-module "computing free evolutions and forced responses of LTI systems"

- finding such signals require knowing a couple of formulas by heart
- partial fraction decomposition is king here, one needs to know how to do that

?