Systems Laboratory, Spring 2025

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building and interpreting phase portraits

Contents map

developed content units	taxonomy levels
phase portrait	u1, e1

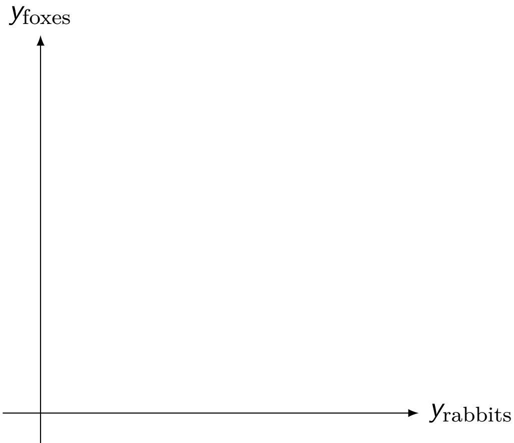
prerequisite content units	taxonomy levels
ODE	u1, e1

Main ILO of sub-module "building and interpreting phase portraits"

Construct and interpret phase portraits of first- and second-order autonomous ODEs using qualitative analysis techniques

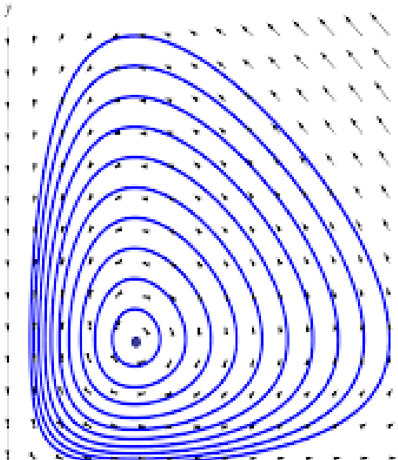
Starting with an example: a Lotka-Volterra model ($\neq$ real world):

$$\begin{cases} \dot{y}_{\text{rabbits}} &= 0.4 \cdot y_{\text{rabbits}} - 0.5 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \\ \dot{y}_{\text{foxes}} &= -3 \cdot y_{\text{foxes}} + 0.7 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \end{cases}$$

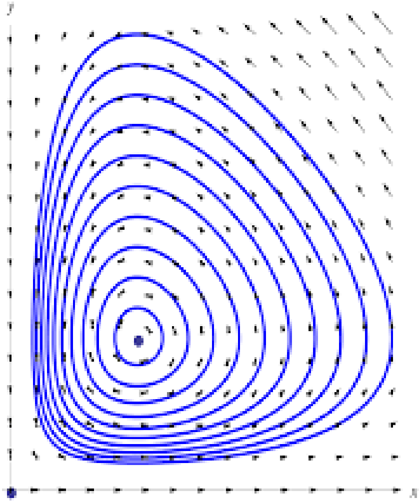


The result, if we were plotting everything

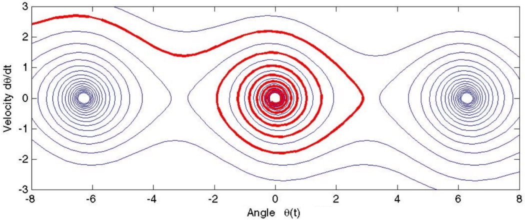
$$\begin{cases} \dot{y}_{\text{rabbits}} &= \quad 0.4 \cdot y_{\text{rabbits}} - 0.5 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \\ \dot{y}_{\text{foxes}} &= -3 \cdot y_{\text{foxes}} + 0.7 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \end{cases}$$



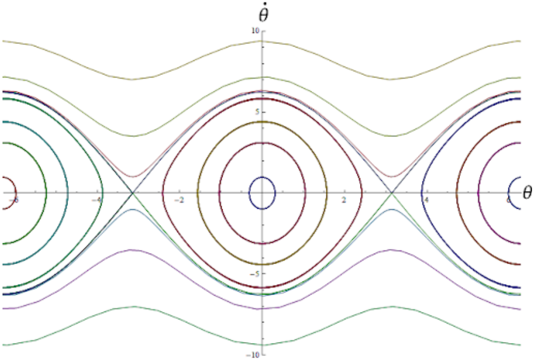
Phase Portrait = a graphical representation of the trajectories of a dynamical system in the state space



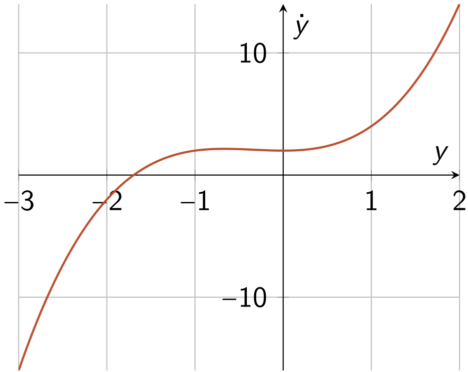
Which system is this one?

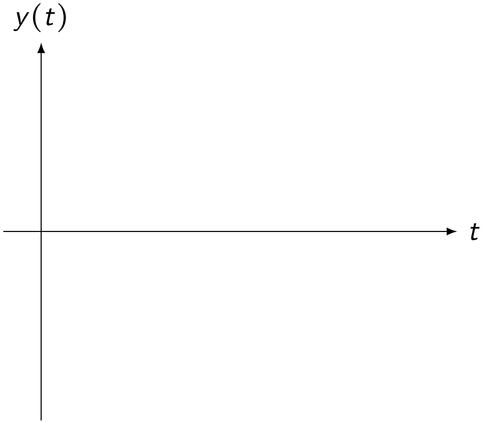


And this one?

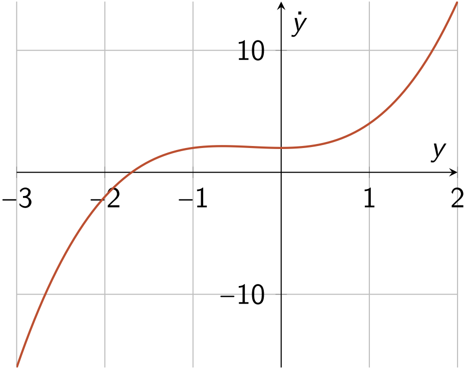


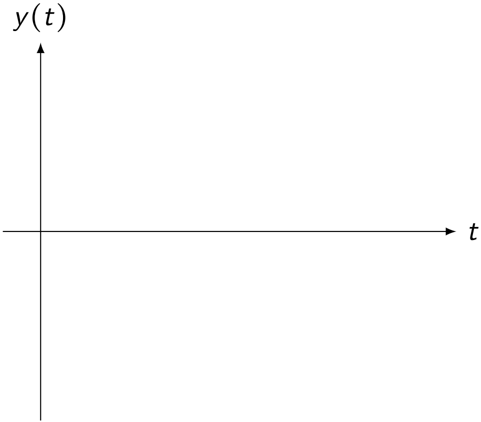
Phase Portraits for first-order ODEs





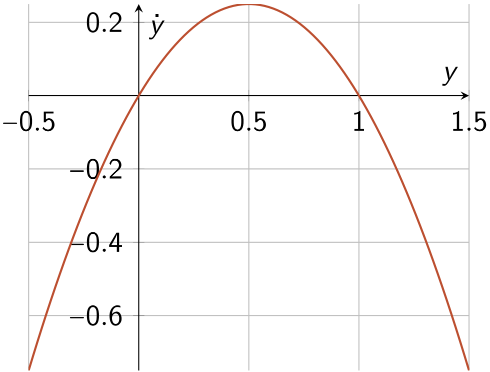
Phase Portraits for first-order ODEs

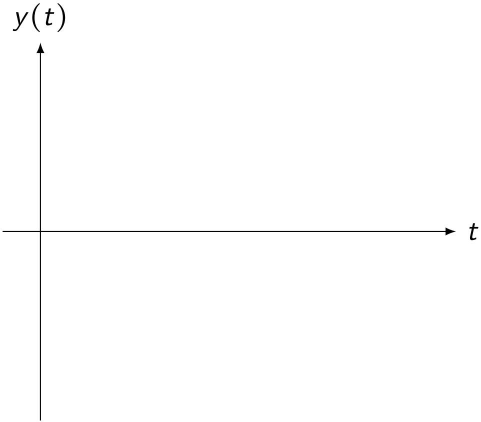




drawing the phase portrait as a 2-D thing in this case is a big error

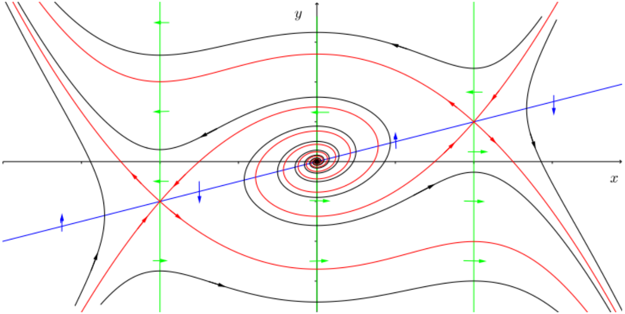
Another example: $\dot{y} = y(1 - y)$





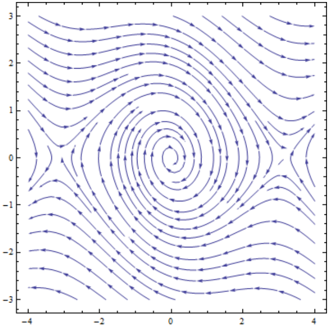
- equilibria: $y = 0$ and $y = 1$
- if $y < 0$ or $y > 1$, $\dot{y}$ is negative (flow left)
- if $0 < y < 1$, $\dot{y}$ is positive (flow right)

Interpreting Phase Portraits



- equilibria: where trajectories do not move
- limit cycles: closed trajectories indicating periodic behavior

Interpreting Phase Portraits



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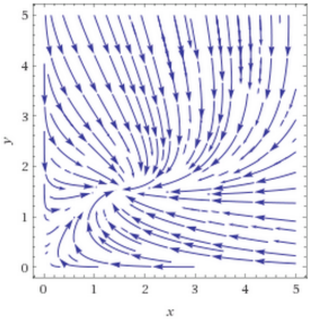
Interpreting Phase Portraits



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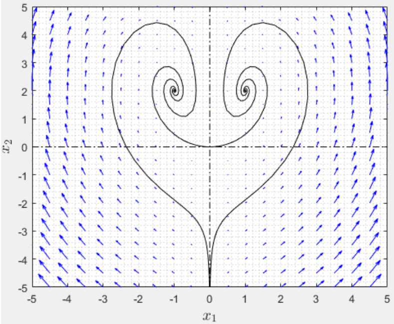
Interpreting Phase Portraits



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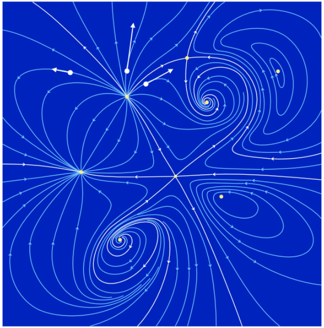
Interpreting Phase Portraits



- equilibria: where trajectories do not move
- limit cycles: closed trajectories indicating periodic behavior

Most important python code for this sub-module

Tutorial on how to plot phase portraits

`https://aleksandarhaber.com/`
`phase-portraits-of-state-space-models-and-differential-equations-in-python`

Self-assessment material

Question 1

What is the primary purpose of a phase portrait?

Potential answers:

I: To find the exact numerical solution of a system
II: To visualize the qualitative behavior of a dynamical system
III: To approximate the integral of a function
IV: To determine the frequency response of a system
V: I do not know

Question 2

How do you determine equilibrium points in a phase portrait of a first-order system $\dot{y} = f(y)$?

Potential answers:

I: By solving $\dot{y} = 0$ for all values of t
II: By solving $f(y) = 0$ for y
III: By integrating $f(y)$ over time
IV: By setting $f(y)$ to a constant value
V: I do not know

Question 3

Which of the following best describes the phase portrait of the system $\dot{y} = y(1 - y)$?

Potential answers:

I: It consists of a single trajectory with no equilibrium points

II: It has two equilibrium points at $y = 0$ and $y = 1$, with flow directions determined by the sign of $f(y)$

III: It has infinitely many equilibrium points

IV: It has no equilibrium points and exhibits oscillatory behavior

V: I do not know

Question 4

What distinguishes the phase portrait of a second-order system from a first-order system?

Potential answers:

I: Second-order phase portraits only have one equilibrium point

II: Second-order phase portraits require a two-dimensional state space (e.g., x vs. $\dot{x}$)

III: First-order systems can have limit cycles, while second-order systems cannot

IV: Phase portraits for second-order systems do not contain information about stability

V: I do not know

Question 5

Which of the following statements about phase portraits of nonlinear systems is correct?

Potential answers:

- I: Nonlinear systems always have a single equilibrium point
- II: Nonlinear phase portraits can be analyzed only by solving the system numerically
- III: Nonlinear phase portraits may exhibit equilibrium points, limit cycles, and chaotic behavior
- IV: Nonlinear phase portraits always resemble those of linear systems for small perturbations
- V: I do not know

Recap of sub-module “building and interpreting phase portraits”

- A phase portrait is a graphical representation of a dynamical systems trajectories in state space.
- Phase portraits provide qualitative insight into system behavior without requiring explicit solutions.
- First-order systems have a one-dimensional state space, while second-order systems require two dimensions, etc.

?