Systems Laboratory, Spring 2025

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Is this function a solution of this ODE?

Contents map

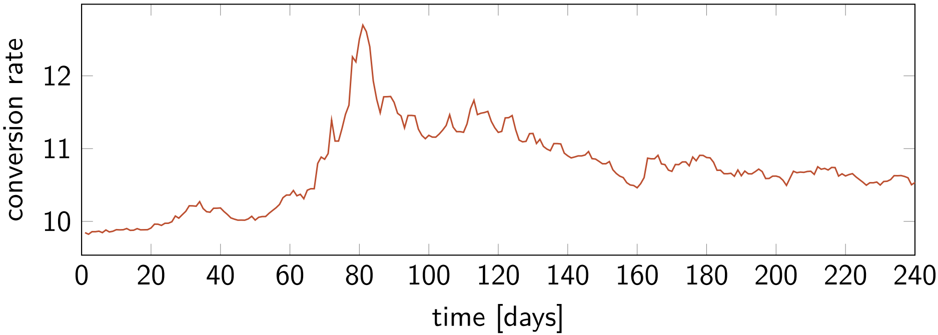
developed content units	taxonomy levels
ODE	u1, e1

prerequisite content units	taxonomy levels
derivative	u1, e1

Main ILO of sub-module "Is this function a solution of this ODE?"

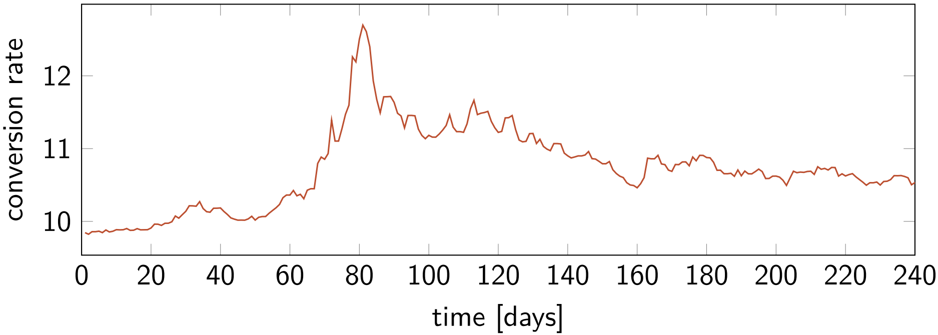
Decide whether a given function is a solution to a specified ODE by direct verification

What is a signal?



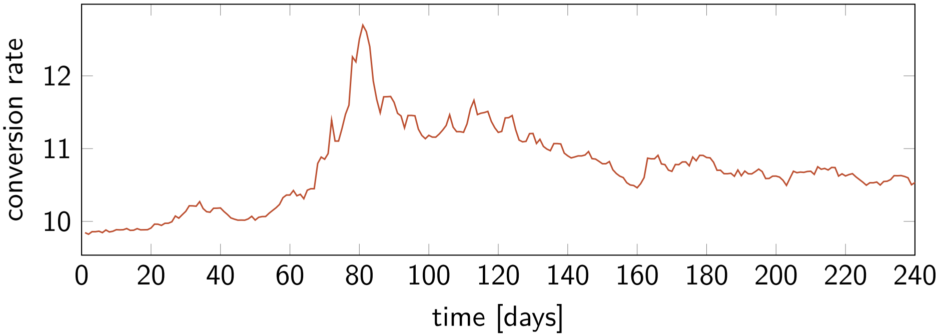
$y(t)$ (assuming t continuous in this module)

What is the derivative of this signal?



$$y(t) \quad \mapsto \quad \frac{dy(t)}{dt} = \dot{y}(t)$$

Would you say that $y(t) = \dot{y}(t)$, in this case?



“uhm, where are we going with all this stuff?”

$\mapsto$ be able to do forecasts

would you be able to compute $y(5)$ from this graph, if you knew that $\dot{y}(t) = y(t)$?

an ODE is a tool to produce forecasts

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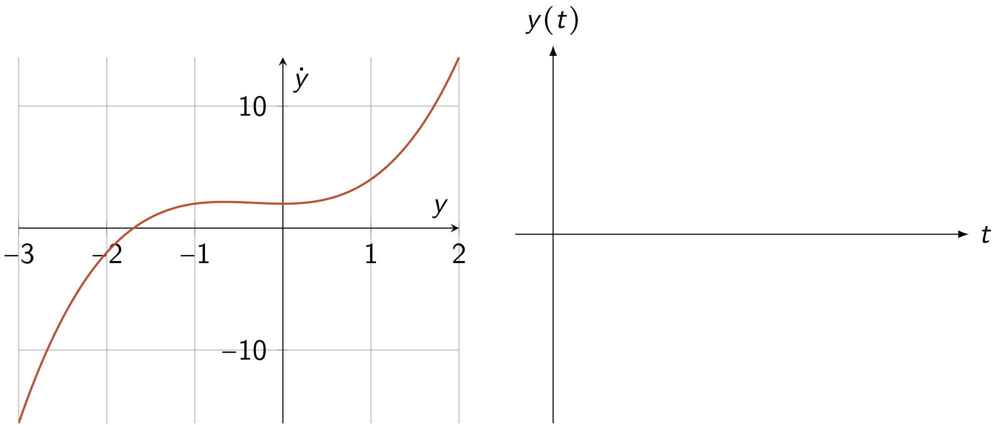
notation: instead of $\dot{y}(t)$ or $y(t)$ we will write $\dot{y}$ or y

But what does it mean to solve an ODE?



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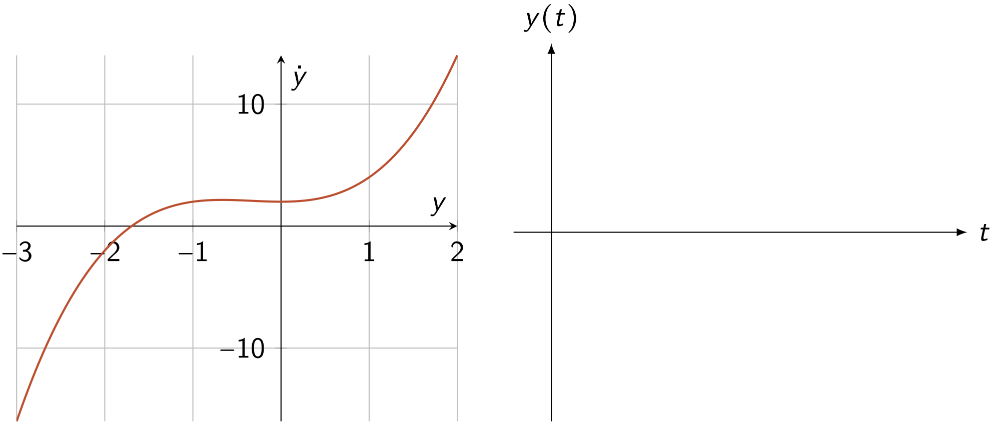

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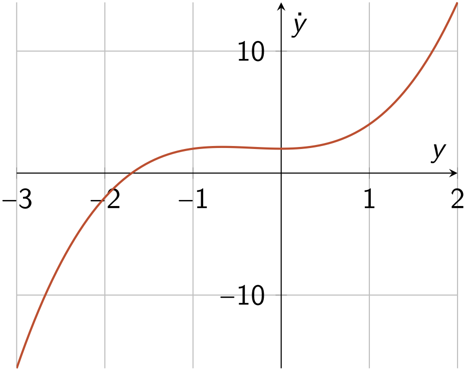
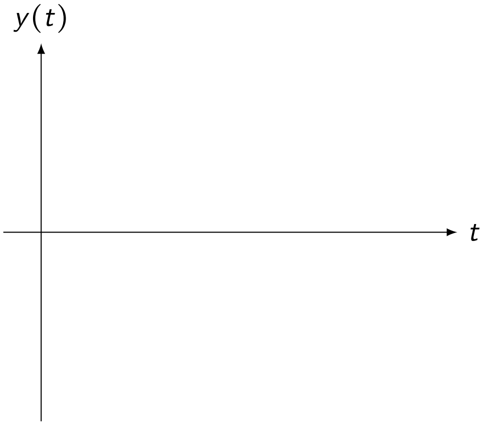
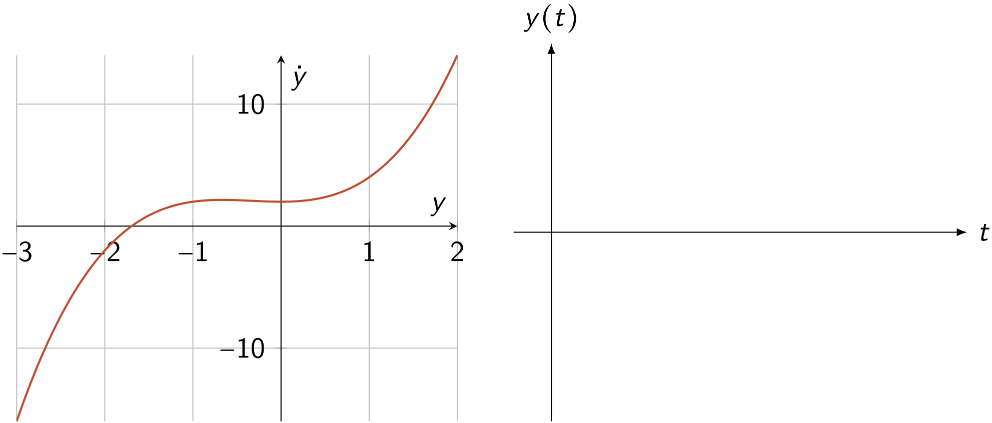
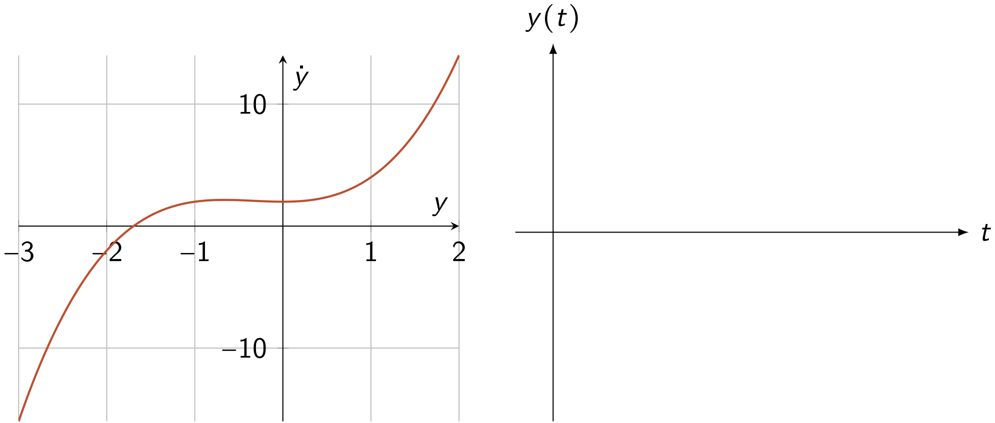




Is knowing the ODE enough to be able to generate a trajectory?



Does $\{y(t) = \cos(t), y(0) = 1\}$ solve this ODE?



Are we done with this?

Decide whether a given function is a solution to a specified ODE by direct verification

→ no, there are still a lot of cases we shall cover

Notation time!

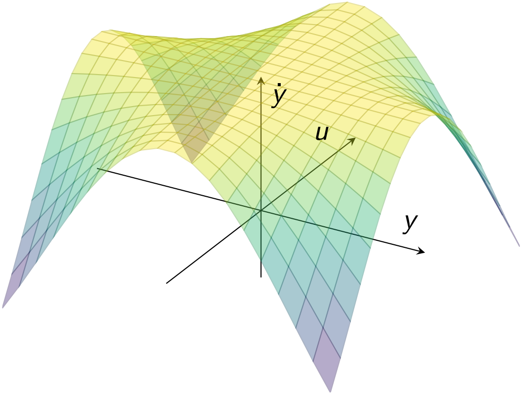
In control, modelling a dynamical system = defining

$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\theta}\right),$$

thus defining:

- the variables
  - $\boldsymbol{u}$ = inputs *(i.e., what we can steer)*
  - $\boldsymbol{d}$ = disturbances *(i.e., what we cannot steer but that still influences the system)*
  - $\boldsymbol{y}$ = outputs *(i.e., what we are interested into)*
- the shape of $\boldsymbol{f}$
- the value of its parameters $\boldsymbol{\theta}$
- bold font = vector

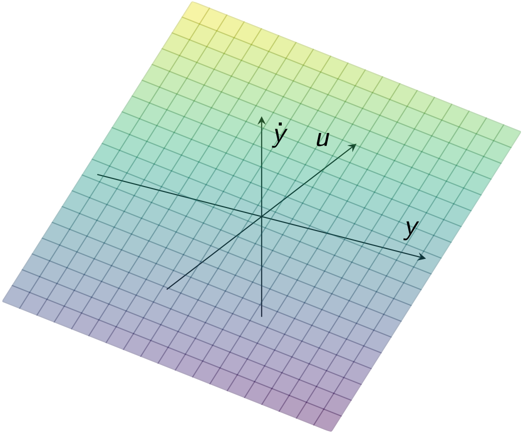
A graphical example of $\dot{y} = f(y, u)$



https://www.geogebra.org/classic/mmppe6hs

A couple of ODEs that you may have already seen, of the type $\dot{y} = ay + bu$

- velocity of a cart: $\dot{v}(t) = -\frac{k}{m} v(t) + \frac{k}{m} F(t)$
- RC-circuits: $\dot{v}_C(t) = -\frac{1}{RC} v_C(t) + \frac{1}{RC} V(t)$



Some more details about the first example

$$\text{notation: } F(t) = ma(t) = m\dot{v}(t) \quad \mapsto \quad F = ma = m\dot{v} \tag{1}$$

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... but $v = \dot{p}$, thus:

$$\begin{cases} \dot{p} = v \\ \dot{v} = \dfrac{F}{m} \end{cases} \tag{2}$$

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this is a system of ODEs

Another practical example

temperature of a small brick in a very large room whose temperature is 20 degrees:

$$\dot{T} = -0.5\left(T - 20\right)$$


Ceci n'est pas une pipe.

Important point: model $\neq$ real world

Ceci n'est pas une brique.

$$\dot{T} = -0.5\left(T - 20\right)$$

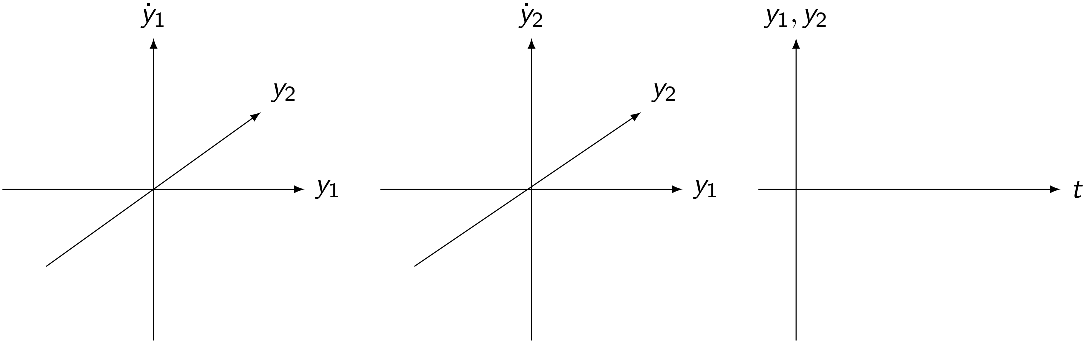
Another example: a Lotka-Volterra model ($\neq$ real world):

$$\begin{cases} \dot{y}_{\text{rabbits}} &= 0.4 \cdot y_{\text{rabbits}} - 0.5 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \\ \dot{y}_{\text{foxes}} &= -3 \cdot y_{\text{foxes}} + 0.7 \cdot y_{\text{rabbits}} \cdot y_{\text{foxes}} \end{cases}$$

What do we mean with "dynamics"? More geometrically

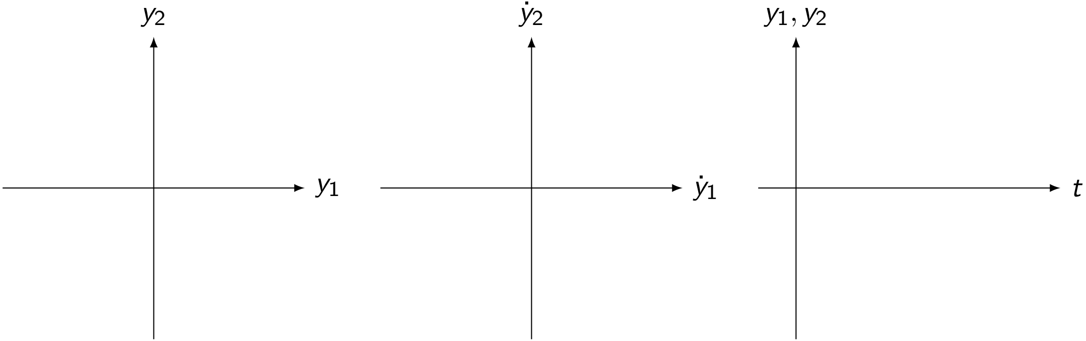
(example: 2D system, autonomous)

example: two dimensional $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y})$ in the sense of $\begin{cases} \dot{y}_1 & = f_1(y_1, y_2) \\ \dot{y}_2 & = f_2(y_1, y_2) \end{cases}$



Same example, alternative viewpoint

$$\begin{cases} \dot{y}_1 & = f_1\left(y_1, y_2\right) \\ \dot{y}_2 & = f_2\left(y_1, y_2\right) \end{cases}$$



Coding the Lotka-Volterra example

$$\begin{cases} \dot{y}_{\text{prey}} &= \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} &= -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

`./LotkaVolterraSimulator.ipynb`

(we'll see later on how this continuous thing is actually implemented in our discrete computers)

Discussion: did we model the Lotka-Volterra dynamical system here?

```
def myModel(y, t):
 #
 # parameters
 alpha = 1.1
 beta  = 0.4
 gamma = 0.4
 delta = 0.1
 #
 # get the individual variables - for readability
 yPrey = y[0]
 yPred = y[1]
 #
 # individual derivatives
 dyPreydt  =   alpha * yPrey - beta  * yPrey * yPred
 dyPreddt  = - gamma * yPred + delta * yPrey * yPred
 #
 return [ dyPreydt, dyPreddt ]
```

Discussion: do we need something more than just the model to simulate the system?

$$\begin{cases} \dot{y}_{\text{prey}} &= \ \ 1.2 y_{\text{prey}} - 0.1 y_{\text{prey}} y_{\text{pred}} \\ \dot{y}_{\text{pred}} &= -0.6 y_{\text{pred}} + 0.2 y_{\text{prey}} y_{\text{pred}} \end{cases}$$

Remember: static $\neq$ dynamic

$$\boldsymbol{y} = \boldsymbol{f}\left(\boldsymbol{u}, \boldsymbol{\theta}\right) \qquad \neq \qquad \dot{\boldsymbol{y}} = \boldsymbol{f}\left(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\theta}\right)$$

Summarizing

Decide whether a given function is a solution to a specified ODE by direct verification

- check y, compute $f(y)$, compute $\dot{y}$
- does $f(y) = \dot{y}$?
- same apply for higher orders / more complex ODES from notational perspectives

Most important python code for this sub-module

Solving ODEs

`https://pythonnumericalmethods.studentorg.berkeley.edu/notebooks/chapter22.06-Python-ODE-Solvers.html`

Self-assessment material

Question 1

Which of the following best describes what it means for a function $y(t)$ to be a solution of an ODE?

Potential answers:

I: It satisfies the ODE for at least one value of t.
II: It satisfies the ODE for all values of t in its domain.
III: It approximately satisfies the ODE within a certain error margin.
IV: It satisfies the ODE only at integer values of t.
V: I do not know

Question 2

What additional information is needed to uniquely determine a solution of an ODE?

Potential answers:

I: The function $y(t)$ itself.
II: An initial condition specifying the value of y at a given time.
III: A boundary condition at two different points.
IV: The highest-order derivative of y.
V: I do not know

Question 3

Given the ODE $\dot{y} = y$, which of the following functions is a solution?

Potential answers:

I: $y(t) = t^2$
II: $y(t) = Ce^t$, where C is a constant.
III: $y(t) = \sin t$
IV: $y(t) = \dfrac{1}{t+1}$
V: I do not know

Question 4

Which of the following differential equations is nonlinear?

Potential answers:

I: $\dot{y} + 2y = 3$
II: $\dot{y} = y^2$
III: $\dot{y} = 3y + 5$
IV: $\dot{y} + \sin y = t$
V: I do not know

Question 5

What is an equilibrium point of the ODE $\dot{y} = y(1-y)$?

Potential answers:

I: $y = 2$

II: $y = 0$ and $y = 1$

III: $y = -1$

IV: $y = \frac{1}{2}$

V: I do not know

Recap of sub-module “Is this function a solution of this ODE?”

- a function is a solution of an ODE if it satisfies the equation for all values in its domain
- initial conditions are necessary to uniquely determine a solution

?