EQUAZIONI DIFFERENZIALI ORDINARIE
CON CONDIZIONI AL CONTORNO

es. $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

$$x \in [a,b]$$

$$\psi(a)=0 \quad e \quad \psi(b)=0$$

Il problema è

$$a(x)\frac{d^2}{dx^2}y(x) + b(x)\frac{d}{dx}y(x) + c(x)y(x) = f(x)$$

$$x \in [a,b]$$

con condizioni al contorno disaccoppiate:

$$\begin{cases} \alpha_0 y(a) + \alpha_1 \frac{d}{dx}y(a) = \lambda_1 \\ \beta_0 y(b) + \beta_1 \frac{d}{dx}y(b) = \lambda_2 \end{cases}$$

ho condizioni di Dirichlet se

$$\begin{cases} y(a) = \alpha \\ y(b) = \beta \end{cases}$$

ho condizioni di Neumann

$$\begin{cases} \frac{d}{dx}y(a) = \alpha \\ \frac{d}{dx}y(b) = \beta \end{cases}$$

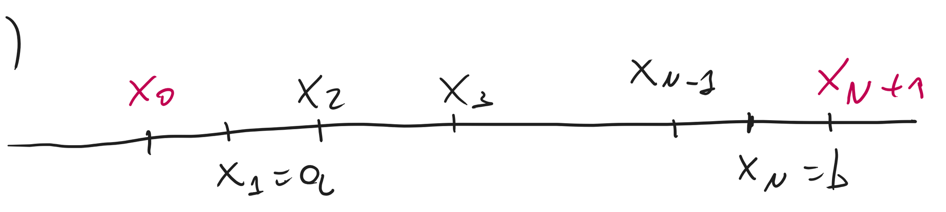
usiamo le differenze finite
griglie equispaziate su $[a,b]$

$$x_1 = a \quad e \quad x_N = b$$

ho $N-1$ intervallini

$$h = \frac{b-a}{N-1}$$

ricordo $y_i = y(x_i)$
e $a_i = a(x_i)$



per $i \in 1,\dots,N-1$

$$\frac{a_i}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{b_i}{2h}(y_{i+1} - y_{i-1}) + c_i y_i = f_i$$

$$\left(\frac{a_i}{h^2} + \frac{b_i}{2h}\right)y_{i+1} + \left(-\frac{2a_i}{h^2} + c_i\right)y_i + \left(\frac{a_i}{h^2} - \frac{b_i}{2h}\right)y_{i-1} = f_i \qquad (I)$$

Per la condizione al bordo in a

$$\alpha_0 y_1 + \frac{\alpha_1}{2h}(y_2 - y_0) = \lambda_1$$

se, caso particolare $\alpha_1 = 0 \Rightarrow y_1 = \frac{\lambda_1}{\alpha_0}$

altrimenti

$$\frac{2h}{\alpha_1}\alpha_0 y_1 + y_2 - y_0 = \frac{2h}{\alpha_1}\lambda_1$$

$$y_0 = y_2 + \frac{2h}{\alpha_1}(\alpha_0 y_1 - \lambda_1)$$

usiamo y_0 in (I) per $i=1$

$$\left(\frac{a_1}{h^2} + \frac{b_1}{2h}\right)y_2 + \left(-\frac{2a_1}{h^2} + c_1\right)y_1 + \left(\frac{a_1}{h^2} - \frac{b_1}{2h}\right)\left(y_2 + \frac{2h}{\alpha_1}(\alpha_0 y_1 - \lambda_1)\right) = f_1$$

$$2\frac{a_1}{h^2}y_2 + \left(-\frac{2a_1}{h^2} + c_1 + \frac{\alpha_0}{\alpha_1}\left(\frac{2a_1}{h} - b_1\right)\right)y_1 = f_1 + \frac{\lambda_1}{\alpha_1}\left(\frac{2a_1}{h} - b_1\right)$$

Scrivo il problema come $(\alpha_1 \neq 0 \text{ e } \beta_1 \neq 0)$

$$\begin{pmatrix} M_{11} & M_{12} & & \\ L_2 & D_2 & U_2 & \\ & L_3 & D_3 & U_3 \\ & & \ddots & \\ & & M_{N-1\,N} & M_{NN} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} f_1 + \frac{\lambda_1}{\alpha_1}\left(\frac{2a_1}{h} - b_1\right) \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N - \frac{\lambda_2}{\beta_1}\left(\frac{2a_N}{h} + b_N\right) \end{pmatrix}$$

con $D_i = \left(-\frac{2a_i}{h^2} + c_i\right)$

$$U_i = \left(\frac{a_i}{h^2} + \frac{b_i}{2h}\right)$$

$$L_i = \left(\frac{a_i}{h^2} - \frac{b_i}{2h}\right)$$

$$M_{11} = \left(-\frac{2a_1}{h^2} + c_1 + \frac{\alpha_0}{\alpha_1}\left(\frac{2a_1}{h} - b_1\right)\right)$$

$$M_{12} = \frac{2a_1}{h}$$

$$M_{NN} = \left(-\frac{2a_N}{h^2} + c_N - \frac{\beta_0}{\beta_1}\left(\frac{2a_N}{h} + b_N\right)\right)$$

$$M_{N-1\,N} = \frac{2a_N}{h^2}$$

se α_1 e β_1 sono $=0$

$$\begin{pmatrix} 1 & & & & \\ L_2 & D_2 & U_2 & & 0 \\ & L_3 & D_3 & U_3 & \\ 0 & & \ddots & & \\ & & L_{N-1} & D_{N-1} & U_{N-1} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda_1/\alpha_0 \\ f_2 \\ \vdots \\ f_{N-1} \\ \lambda_2/\beta_0 \end{pmatrix}}_{\vec{f}}$$

Formalmente $M\vec{y} = \vec{f}$

$$\vec{y} = M^{-1}\vec{f}$$