Birapporto: rapporto tra rapporti

birapporto:

① $|CA| = \sin\frac{\alpha}{2}\cdot\frac{|BC|}{\sin\alpha}$; $|CB| = \sin\frac{\beta}{2}\cdot\frac{|AC|}{\sin\beta}$ $\Rightarrow \frac{|CA|}{|CB|} = \frac{\sin\alpha}{\sin\beta}\cdot\frac{\sin\frac{\beta}{2}}{\sin\frac{\alpha}{2}}$

② $|DA| = \sin\frac{\alpha}{2}\cdot\frac{|BA|}{\sin\alpha}$; $|DB| = \sin\frac{\beta}{2}\cdot\frac{|BA|}{\sin\beta}$ $\Rightarrow \frac{|DA|}{|DB|} = \frac{\sin\alpha}{\sin\beta}\cdot\frac{\sin\frac{\beta}{2}}{\sin\frac{\alpha}{2}}$

$\Rightarrow \frac{|CA|/|BC|}{|DA|/|DB|} = \frac{\sin\alpha}{\sin\beta}\cdot\frac{\sin\frac{\beta}{2}}{\sin\frac{\alpha}{2}}\cdot\frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} = \frac{\sin\alpha}{\sin\beta}\cdot\frac{\sin\frac{\beta}{2}}{\sin\frac{\alpha}{2}}$

Ulteriormente questi oggetti vengono studiati: α,β, ... sono chiamate misure, lunghezze, ...

Birapporto = rapporto tra rapporti

$\frac{a-c}{b-c} : \frac{a-d}{b-d} = \frac{c-a}{c-b}\cdot\frac{b-d}{d-a}$

Birapporti

definizione
calcolo
proprietà (invarianza, particolarità)

Definizione A,B,C,D punti allineati e γ un 3 diverso ordinatamente su $\mathbb{P}^1(K)$.
ABC danno coordinate sulla retta come $A=0$, $B=\infty$, $C=1$ e allora $(ABCD)$ è la coordinata di D nel riferimento A,B,C.

$$\text{Se } A=\begin{pmatrix}a\\c\end{pmatrix},\ B=\begin{pmatrix}b\\d\end{pmatrix},\ C=\begin{pmatrix}e\\g\end{pmatrix},\ D=\begin{pmatrix}f\\h\end{pmatrix}$$

allora $(ABCD) = \frac{eh-gf}{ah-cf}\cdot\frac{ah}{eh-gf} = \frac{ah}{ah-cf} = \frac{a/c}{a/c-f/h}$ $\quad a=1,c=0,e=1,g=0$

tutti $(ABCD) = \begin{cases} \text{se } ha\neq cb & 0\\ \text{se } a=b=0 & \infty\\ \text{se } a=b\neq 0 & 1\end{cases}$

Elementi non caso $\mathbb{P}^1(K)$:

$$A=\begin{pmatrix}1\\0\end{pmatrix},\ B=\begin{pmatrix}0\\1\end{pmatrix},\ C=\begin{pmatrix}1\\1\end{pmatrix},\ D=\begin{pmatrix}x\\y\end{pmatrix}$$

$$(ABCD) = \frac{x}{y} \quad \text{dove } \begin{pmatrix}x\\y\end{pmatrix}: \lambda A+\mu B = \begin{pmatrix}\lambda\\0\end{pmatrix}+\begin{pmatrix}0\\\mu\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$$

$$\text{dove } \begin{cases}\lambda=x\\ \mu=y\end{cases}$$

$$\text{e } \lambda\begin{pmatrix}a\\c\end{pmatrix}+\mu\begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}e\\g\end{pmatrix}$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\lambda\\ \mu\end{pmatrix} = \begin{pmatrix}e\\ g\end{pmatrix}$$

$$\lambda = \frac{\begin{vmatrix}e & b\\ g & d\end{vmatrix}}{\begin{vmatrix}a & b\\ c & d\end{vmatrix}} = \frac{eh-gf}{ah-cf}$$

$$\mu = \frac{\begin{vmatrix}a & e\\ c & g\end{vmatrix}}{\begin{vmatrix}a & b\\ c & d\end{vmatrix}} = \frac{ag-ce}{ah-cf}$$

$$\text{e } x = \lambda\begin{pmatrix}a\\c\end{pmatrix} = \frac{eh-gf}{ah-cf}\begin{pmatrix}a\\c\end{pmatrix}, \quad y = \mu\begin{pmatrix}b\\d\end{pmatrix} = \frac{ag-ce}{ah-cf}\begin{pmatrix}b\\d\end{pmatrix}$$

Scelta $\forall\, t,t'$:

$$x_t\begin{pmatrix}ta\\c(t+1)\end{pmatrix} + y_t\begin{pmatrix}tb\\d(t+1)\end{pmatrix} = \begin{pmatrix}e\\g(t+1)\end{pmatrix}$$

$$\begin{pmatrix}ta & tb\\ c(t+1) & d(t+1)\end{pmatrix}\begin{pmatrix}x_t\\ y_t\end{pmatrix} = \begin{pmatrix}e\\ g(t+1)\end{pmatrix}$$

$$x_t = \begin{vmatrix}e & tb\\ g(t+1) & d(t+1)\end{vmatrix} \cdot \frac{1}{\begin{vmatrix}ta & tb\\ c(t+1) & d(t+1)\end{vmatrix}} = \frac{edt(t+1)-bgt(t+1)}{adt^2(t+1)-bct^2(t+1)}$$

$$y_t = \begin{vmatrix}ta & e\\ c(t+1) & g(t+1)\end{vmatrix} \cdot \frac{1}{\begin{vmatrix}ta & tb\\ c(t+1) & d(t+1)\end{vmatrix}} = \frac{agt(t+1)-cet(t+1)}{adt^2(t+1)-bct^2(t+1)}$$

$$x = \begin{vmatrix}ta & tb\\ ta+c & td+d\end{vmatrix} \cdot \frac{1}{\begin{vmatrix}ta & tb\\ c & d\end{vmatrix}} = \frac{adt^2+adt-bct^2-bct}{adt-bc}$$

$$y = \begin{vmatrix}ta & e\\ ta+c & g\end{vmatrix} \cdot \frac{1}{\begin{vmatrix}ta & tb\\ c & d\end{vmatrix}} = \frac{agt+ag-cet-ce}{adt-bc}$$

$$(ABCD) = \frac{x}{y} = \frac{bd}{(a-c)(b-d)}\cdot\frac{t+\frac{c}{a}}{t+\frac{d}{b}} = \frac{b-d}{a-c}\cdot\frac{c-a}{b-d}$$

$$(ABCD) = \frac{c-a}{c-b}\cdot\frac{b-d}{a-d}$$

Indici (a,b,c,d): $\frac{c-a}{c-b}\cdot\frac{a-b}{a-d}$ tutte uguali tra loro se a,b,c,d sono tutti diversi

Esercizio: $A=\begin{pmatrix}a\\c\end{pmatrix}$, $B=\begin{pmatrix}b\\d\end{pmatrix}$, $C=\begin{pmatrix}e\\g\end{pmatrix}$, $D=\begin{pmatrix}f\\h\end{pmatrix}$?

$$(ABCD) = \frac{\begin{vmatrix}e & b\\ g & d\end{vmatrix}}{\begin{vmatrix}e & f\\ g & h\end{vmatrix}}\cdot\frac{\begin{vmatrix}a & f\\ c & h\end{vmatrix}}{\begin{vmatrix}a & b\\ c & d\end{vmatrix}}$$

Esempio

$$(ABCD) = \left(\begin{pmatrix}3\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix}\right) = \left(\frac{5}{3}\ \frac{3}{2}\ \frac{2}{1}\ \frac{1}{1}\right) = \frac{1}{2}\cdot\frac{3}{1} = \frac{3}{2}$$

Invece più "ordinato" (e consigliato) per calcolare birapporti.

Rette proiettive

$$(a:b:1)\cdot\lambda = (a:b:\lambda)$$

$$(a:b:\infty)\cdot\lambda = (a:b:\infty)$$

Proprietà Birapporto: $\mathbb{P}^1(K)\to\mathbb{P}^1(K)$ biiettiva

$$(ABCD) = (BADC) = (CDAB) = (DCBA)$$

$$(ABCD) = \frac{1}{(ACBD)} = 1-(ADBC)$$

Se $(ABCD)=-1$ si dice che A,B separano armonicamente C,D e viceversa.

Se $(ABCD)=\lambda$ allora $(ABDC)=\frac{1}{\lambda}$, $(ACBD)=1-\lambda$, $(ACDB)=\frac{\lambda-1}{\lambda}$, $(ADBC)=\frac{1}{1-\lambda}$, $(ADCB)=\frac{\lambda}{\lambda-1}$

Se $\lambda=-1$ o $\lambda=\frac{1}{2}$ o $\lambda=2$ si dice che A,B,C,D è una quaterna armonica.

Teorema Dato $\lambda\neq 0,1$ esistono 4 punti A,B,C,D di $\mathbb{P}^1(K)$ tali che $(ABCD)=\lambda$.

Dimostrazione: $A=0$, $B=\infty$, $C=1$, $D=\lambda$.

Teorema Siano A,B,C punti distinti di $\mathbb{P}^1(K)$ e $\lambda\neq 0,1$. Allora esiste un unico $D\in\mathbb{P}^1(K)$ tale che $(ABCD)=\lambda$.

Dimostrazione: $A=0$, $B=\infty$, $C=1$, $D=\lambda$.

Proposizione Siano A,B,C,D punti distinti di $\mathbb{P}^1(K)$. Allora $(ABCD)=(CDAB)$ se e solo se A,B,C,D sono a due a due distinti.

Dimostrazione: Se $A=0$, $B=\infty$, $C=1$, $D=\lambda$ allora $(ABCD)=\lambda=(CDAB)$.

Se $A=\begin{pmatrix}a\\c\end{pmatrix}$, $B=\begin{pmatrix}b\\d\end{pmatrix}$, $C=\begin{pmatrix}e\\g\end{pmatrix}$, $D=\begin{pmatrix}f\\h\end{pmatrix}$ allora $(ABCD)=\frac{eh-gf}{ah-cf}\cdot\frac{ah}{eh-gf}=\frac{ah}{ah-cf}$

$(CDAB)=\frac{ah}{ah-ce}$ $\quad$ $(ABCD)=(CDAB)\iff ah-cf=ah-ce \iff cf=ce$

$\iff c=0$ o $e=f$ $\iff$ $A=B$ o $C=D$

Esercizio: Siano A,B,C,D punti allineati su $\mathbb{P}^1$ e γ un 3 diverso da A,B,C.

Allora esiste un unico $\varphi\in PGL(2,K)$ tale che $\varphi(A)=\gamma$, $\varphi(B)=\infty$, $\varphi(C)=1$ e $(ABCD)=(\varphi(A)\varphi(B)\varphi(C)\varphi(D))$.

Dimostrazione: Se $A=0$, $B=\infty$, $C=1$ allora $(ABCD)=D$ e $\varphi=\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}$ va bene.

Se $A=\begin{pmatrix}a\\c\end{pmatrix}$, $B=\begin{pmatrix}b\\d\end{pmatrix}$, $C=\begin{pmatrix}e\\g\end{pmatrix}$ allora $\varphi=\begin{pmatrix}\gamma eb-ga & \gamma(ea-cb)\\ gb-ed & ea-cb\end{pmatrix}$ va bene.