Ricordo: $\forall v \neq 0$ vale esattamente $\mathbb{K}\langle v \rangle$ + $\mathbb{K}$-sp vett di dim 1 + $\mathbb{K}$-sp vett di dim 1 + $\mathbb{K}$-sp vett di dim 1

$\mathbb{K} = \mathbb{C}$? → garantito ALGEBRICAMENTE

se conosco $\det = \lambda^n w$ allora $\left(\frac{1}{\lambda}\right)^n \det(\frac{1}{\lambda} v) = \det v$

$\Rightarrow \det(\frac{1}{\lambda} v) = \frac{1}{\lambda^n} \det v$

$\det(\lambda v) = \lambda^n \det v$

Def Gruppo lineare: $V \circlearrowright V(\mathbb{C}) = \mathbb{C}^n$

Gruppo lineare $\mathrm{GL}(V) = \{ f : V \to V \text{ lineare} \mid f \text{ è un isomorfismo} \}$

$(V, +) \longrightarrow (V, +) = \mathbb{K}^n \mathbb{K}^n$

$\begin{pmatrix} x \\ y \end{pmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det = \frac{a}{c} \frac{b}{d} \in \mathbb{K}^*$

Corollario: $\mathrm{GL}_n(\mathbb{K}) = \{ A \in \mathrm{M}_n(\mathbb{K}) \mid \det A \neq 0 \}$

Prodotto:

- $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ $\quad \mathbb{K} = \mathbb{R}, \mathbb{C}$
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$ $\quad \mathbb{K} = \mathbb{R}$
- $\langle v, \lambda w \rangle = \lambda \langle v, w \rangle$ $\quad \mathbb{K} = \mathbb{C}$
- $\langle v, w + z \rangle = \langle v, w \rangle + \langle v, z \rangle$
- $\langle v, v \rangle = 0 \iff v = 0$

$\mathbb{K} = \mathbb{C}$: $\langle \lambda v, \lambda v \rangle = \lambda \overline{\lambda} \langle v, v \rangle = |\lambda|^2 \langle v, v \rangle$

$\mathbb{K} = \mathbb{R}$: $\langle \lambda v, \lambda v \rangle = \lambda^2 \langle v, v \rangle$

Standard: $\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$

Standard hermitiano: $\langle x, y \rangle = x_1 \overline{y_1} + \dots + x_n \overline{y_n}$

Standard di Minkowski: $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$

Standard di Lorentz: $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - c^2 x_4 y_4$

- Norma: $\|v\| = \sqrt{\langle v, v \rangle}$ $\quad$ $\|v\| \in \mathbb{R}$ anche se $\mathbb{K} = \mathbb{C}$
- $\|v\| = 0 \iff v = 0$
- $\|\lambda v\| = |\lambda| \, \|v\|$
- $\|v + w\| \leq \|v\| + \|w\|$ $\quad$ disuguaglianza triangolare (dimostrata sotto)

Def: $\mathrm{U}_n(\mathbb{C}) = \{ A \in \mathrm{M}_n(\mathbb{C}) : A^{\mathrm{T}} \overline{A} = \mathrm{Id} \} \subseteq \mathrm{GL}_n(\mathbb{C})$

$\mathrm{U}_n(\mathbb{C})$ = gruppo unitario = matrici unitarie

se $A \in \mathrm{U}_n(\mathbb{C})$ allora:

$|\det(A)| = 1$ $\quad$ $A^{\mathrm{T}} \overline{A} = \mathrm{Id}$

$\det(A) \neq 0$ $\quad$ $\det(\overline{A}^{\mathrm{T}} A) = 1$

$\det(A^{\mathrm{T}}) \det(\overline{A}) = 1$

$\det(A) \det(\overline{A}) = 1$

$\det(A) \overline{\det(A)} = 1$

$|\det(A)|^2 = 1$

autovalori di A sono tutti in modulo $= 1$ anche se $\mathbb{K} = \mathbb{C}$

$\|Av\|^2 = \|v\|^2$

$\langle Av, Aw \rangle = \langle v, w \rangle$

$\forall v \neq 0 \quad \frac{\|Av\|}{\|v\|} = 1$

$\|Av\| = \|v\|$

(poi interpreteremo sta proprietà come isometria, e vedremo che $\mathrm{U}_n(\mathbb{C})$ = isometrie)

$\mathrm{SL}_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ $\xrightarrow{\det}$ $\mathbb{C}^*$

$\mathrm{SL}_n(\mathbb{C}) \subseteq \mathrm{U}_n(\mathbb{C})$ $\xrightarrow{\det}$ $\mathrm{U}_1(\mathbb{C}) = \{ |z| = 1 \}$

$\det|_{\mathrm{U}_n(\mathbb{C})} : \mathrm{U}_n(\mathbb{C}) \to \mathrm{U}_1(\mathbb{C})$ $\quad$ suriettiva (vedremo)

$\pi : \mathrm{U}_n(\mathbb{C}) \to \mathbb{R}$ $\quad$ $\pi(A) = \frac{1}{2\pi} \arg(\det A)$

$\mathrm{SU}_n(\mathbb{C}) = \{ A \in \mathrm{U}_n(\mathbb{C}) : \det A = 1 \}$

$\mathrm{SU}_n(\mathbb{C}) = \ker(\det|_{\mathrm{U}_n(\mathbb{C})})$

$\mathrm{SU}_n(\mathbb{C}) = \ker \pi$

$\mathrm{SU}_n(\mathbb{C}) \subseteq \mathrm{U}_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$

$\mathrm{SU}_n(\mathbb{C}) \subseteq \mathrm{SL}_n(\mathbb{C})$

$\mathrm{SU}_1(\mathbb{C}) = \{ (a) : |a| = 1 \} \cong \mathrm{U}_1(\mathbb{C})$

$\mathrm{SU}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$

$\mathrm{SU}_2(\mathbb{C}) \cong \mathbb{S}^3$

$\mathrm{U}_2(\mathbb{C}) \cong \mathbb{S}^3 \times \mathbb{S}^1$

$\mathrm{SU}_3(\mathbb{C}) \cong \mathbb{S}^5 \times \mathbb{S}^3$

$\mathrm{U}_3(\mathbb{C}) \cong \mathbb{S}^5 \times \mathbb{S}^3 \times \mathbb{S}^1$

Teorema di Cauchy-Binet: $A, B \in \mathrm{M}_n(\mathbb{K})$

$\det(AB) = \det(A) \det(B)$

$\det(AB) = \det(A) \det(B) = \det(A) \det(B) = \det(BA)$

$\det(AB) = \det(BA)$

Prima: $\det(AB) = \det(A) \det(B) \Rightarrow \det B \det(A) = \det A \det(B)$

Come fare $\det = 0$...

Teorema spettrale autoaggiunti (R):

def: S è un endo autoaggiunto se $S = S^{\mathrm{T}}$

$\det(A - \lambda \mathrm{Id}) = 0$ $\quad$ $\det(A - \lambda \mathrm{Id}) = 0