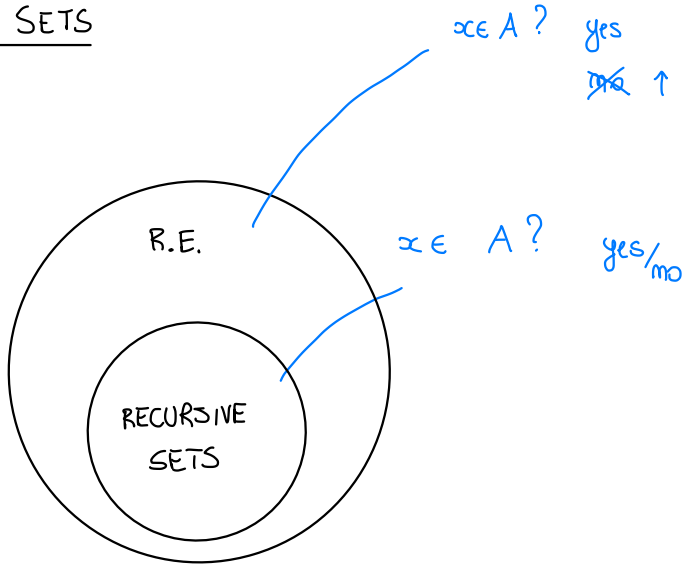
COMPUTABILITY (24/11/2026)

RECURSIVELY ENUMERABLE SETS



Def. (r.e. set) $A \subseteq \mathbb{N}$ is recursively enumerable (r.e.) if the semi-characteristic function $sc_A : \mathbb{N} \to \mathbb{N}$

$$sc_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

is computable

A property $Q(\vec{x}) \subseteq \mathbb{N}^k$ is semi-decidable if $sc_Q : \mathbb{N}^k \to \mathbb{N}$

$$sc_Q(\vec{x}) = \begin{cases} 1 & \text{if } Q(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases}$$

computable

OBSERVATION: if $Q(x) \subseteq \mathbb{N}$

$Q(x)$ is semi-decidable iff $Q(x) \subseteq A$ r.e.

$= \{x \mid Q(x)\}$

(recursive / r.e. sets $A \subseteq \mathbb{N}^k$ could be defined as well)

OBSERVATION: Let $A \subseteq \mathbb{N}$ be a set

A recursive $\iff$ A, $\bar{A}$ r.e.

proof

($\Rightarrow$) Let $A \subseteq \mathbb{N}$ be recursive, i.e. $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

is computable.

We want to show that

$$sc_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{if } x \notin A \end{cases}$$ computable

Intuitively, if P_{χ_A} is a program for χ_A then

def $P_{sc_A}(x)$:

$y := P_{\chi_A}(x)$

if $y == 1$

return 1

else loop

formally

$$sc_A(x) = (\mu w.\ \overline{sg}(\chi_A(x))) + 1$$

χ_A: 1 if $x \in A$, ↑ otherwise

$\overline{sg}$: 1 if $x \in A$, 0 otherwise

$\mu w.$: 0 if $x \in A$, 1 otherwise

$+1$: 0 if $x \in A$, ↑ otherwise

$\mu y.\ f(\vec{x}, y)$

computable (composition / minimalisation of computable function)

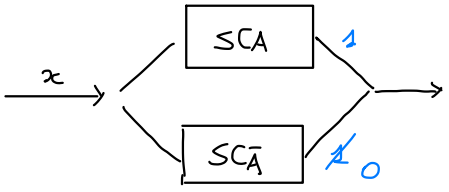
For $\bar{A}$, observe $\bar{A}$ is recursive, hence we conclude $\bar{A}$ r.e. from the first part.

($\Leftarrow$) Let $A, \bar{A}$ r.e., i.e. the semi-characteristic functions

$$sc_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

$$1 \dot{-} sc_{\bar{A}}(x) = \begin{cases} 0 & \text{if } x \notin A \\ \uparrow & \text{otherwise} \end{cases}$$

computable

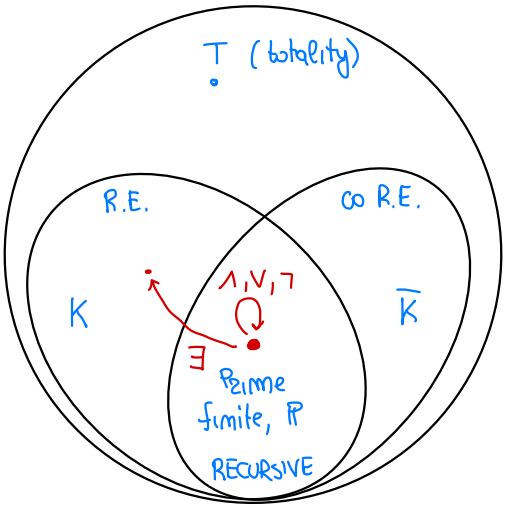


let $e_1, e_0 \in \mathbb{N}$ be s.t. $\varphi_{e_1} = sc_A$ $\varphi_{e_0} = 1 \dot{-} sc_{\bar{A}}$

then

$$\chi_A(x) = \left(\mu(\sigma, t) \,.\, S(e_1, x, \sigma, t) \vee S(e_0, x, \sigma, t) \right)_\sigma$$
$$= \left(\mu\omega \,.\, S(e_1, x, (\omega)_1, (\omega)_2) \vee S(e_0, x, (\omega)_1, (\omega)_2) \right)_1$$

computable, hence A recursive. □



co r.e. set = sets whose complement is r.e.

* K is not recursive

but it is r.e.

$$sc_K(x) = \mathbb{1}\left(\varphi_x(x)\right)$$
$$= \mathbb{1}\left(\varphi_{-v}(x, x)\right)$$

computable

* Existential quantification

$Q(t, \vec{x}) \subseteq \mathbb{N}^{k+1}$ decidable

$\rightsquigarrow$ $\exists t.\, Q(t, \vec{x})$ semi-decidable

* Structure Theorem

Let $P(\vec{x}) \subseteq \mathbb{N}^k$ be a predicate

$P(\vec{x}) \subseteq \mathbb{N}^k$ semi-decidable $\iff$ there is $Q(t, \vec{x}) \subseteq \mathbb{N}^{k+1}$ decidable

s.t. $P(\vec{x}) \equiv \exists t.\, Q(t, \vec{x})$

proof

($\Rightarrow$) Let $P(\vec{x}) \subseteq \mathbb{N}^k$ semi-decidable, i.e.

$$sc_P(\vec{x}) = \begin{cases} 1 & \text{if } P(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases} \qquad \text{computable}$$

i.e. there is $e \in \mathbb{N}$ s.t. $\varphi_e^{(k)} = sc_P$

Observe $P(\vec{x})$ iff $sc_P(x) = 1$ iff $\exists t.\ S^{(k)}(e, \vec{x}, 1, t)$

If we define

$$Q(t, \vec{x}) \equiv S^{(k)}(e, \vec{x}, 1, t) \qquad \text{decidable}$$

and

$$P(\vec{x}) \equiv \exists t.\ Q(t, \vec{x}) \qquad \text{as desired.}$$

($\Leftarrow$) We assume $P(\vec{x}) \equiv \exists t. Q(t, \vec{x})$ with $Q(t, \vec{x})$ decidable

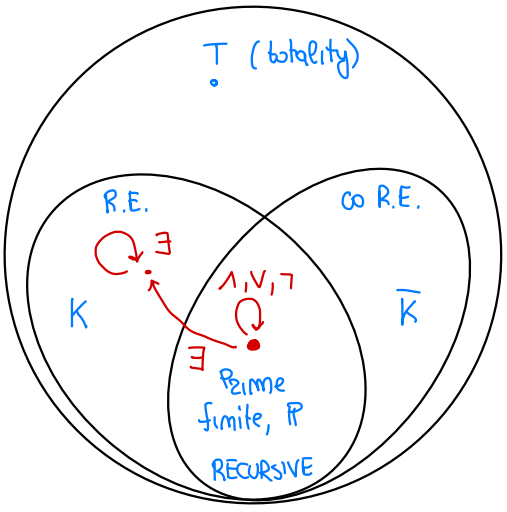
$$sc_P(\vec{x}) = \begin{cases} 1 & \text{if } P(\vec{x}) \leftrightarrow \exists t.\ Q(t, \vec{x}) \leftrightarrow \exists t.\ \chi_Q(t, \vec{x}) = 1 \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mathbb{1}\left(\mu t.\ |\chi_Q(t, \vec{x}) - 1| \right) \qquad \text{computable}$$

(χ_Q: computable)

□

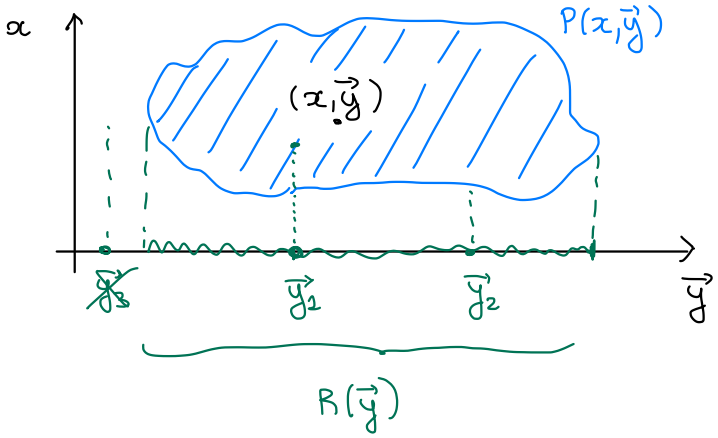
What if I take on semidecidable properties a further existential quantification?



Projection Theorem :

Let $P(x,\vec{y}) \subseteq \mathbb{N}^{k+1}$ semi-decidable

Then also $R(\vec{y}) \equiv \exists x.\, P(x,\vec{y}) \subseteq \mathbb{N}^k$ is semi-decidable



proof

Let $P(x,\vec{y}) \subseteq \mathbb{N}^{k+1}$ semi-decidable. By the structure theorem

$$P(x,\vec{y}) \equiv \exists t.\, Q(t,x,\vec{y}) \quad \text{with } Q(t,x,\vec{y}) \text{ decidable}$$

Then

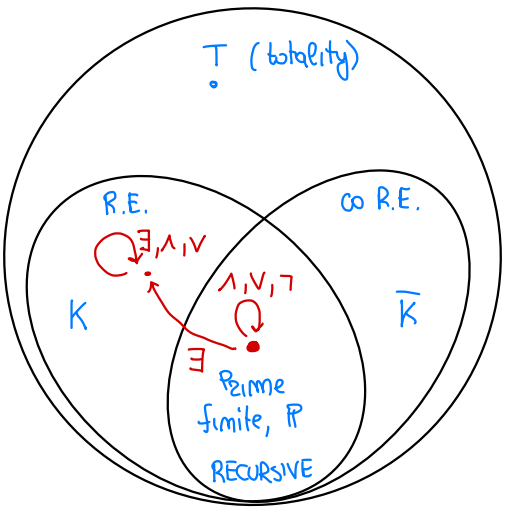
$$R(\vec{y}) \equiv \exists x.\, P(x,\vec{y}) \equiv \exists x.\, \exists t.\, Q(t,x,\vec{y})$$

$$\equiv \exists w.\, Q((w)_1,(w)_2,\vec{y})$$

$$Q'(w,\vec{y}) \equiv Q((w)_1,(w)_2,\vec{y}) \text{ decidable}$$

$$\equiv \exists w.\, Q'(w,\vec{y})$$

By the structure theorem $R(\vec{y})$ is semi-decidable □



What about conjunction / disjunction of semi-decidable predicates?

OBSERVATION: Let $P(\vec{x}), Q(\vec{x}) \subseteq \mathbb{N}^k$ be semi-decidable

① $P(\vec{x}) \wedge Q(\vec{x})$

② $P(\vec{x}) \vee Q(\vec{x})$

are semi-decidable

proof

By the structure theorem

$$P(\vec{x}) \equiv \exists t.\ P'(t, \vec{x})$$

$$Q(\vec{x}) \equiv \exists t.\ Q'(t, \vec{x})$$

$P'(t,\vec{x}), Q'(t,\vec{x})$ decidable

Then

① $$P(\vec{x}) \wedge Q(\vec{x}) \equiv \exists t_1.\ P'(t_1, \vec{x}) \wedge \exists t_2.\ Q'(t_2, \vec{x})$$

$$\equiv \exists (t_1, t_2).\ \left(P'(t_1, \vec{x}) \wedge Q'(t_2, \vec{x})\right)$$

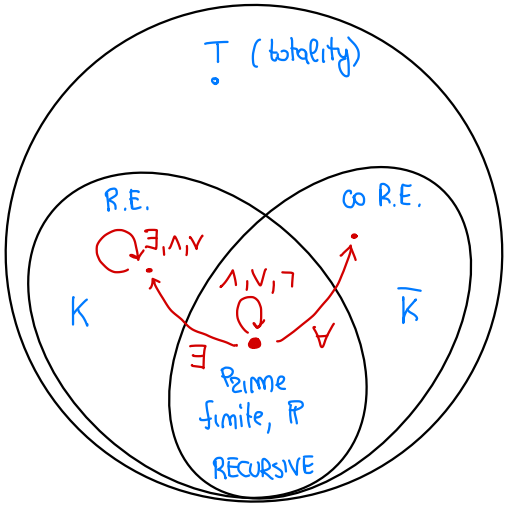
$$\equiv \exists \omega.\ \underbrace{P'((\omega)_1, \vec{x}) \wedge Q'((\omega)_2, \vec{x})}_{\text{decidable}}$$

hence by the structure theorem $P(\vec{x}) \wedge Q(\vec{x})$ semidecidable

② $$P(\vec{x}) \vee Q(\vec{x}) \equiv \exists t.\ P'(t, \vec{x}) \vee \exists t.\ Q'(t, \vec{x})$$

$$\equiv \exists t.\ \underbrace{\left(P'(t, \vec{x}) \vee Q'(t, \vec{x})\right)}_{\text{decidable}}$$

hence by the structure theorem $P(\vec{x}) \vee Q(\vec{x})$ semi-decidable $\square$



* Negation

$Q(x) \equiv$ "$x \in K$" semi-decidable

but

$\neg Q(x) \equiv$ "$x \notin K$"

not semi-decidable

Universal Quantification

$R(t,x) \equiv \neg H(x,x,t)$ decidable

but

$$\forall t.\ R(t,x) \equiv \forall t.\ \neg H(x,x,t) \equiv x \notin K$$

not semi-decidable.

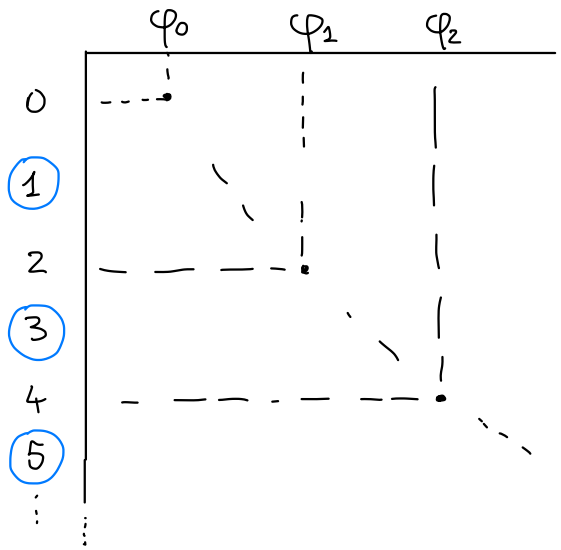
EXERCISE: Given $P(x,\vec{y}) \subseteq \mathbb{N}^k$. Is it true that

$\exists x.\ P(x,\vec{y})$ semi-decidable $\Rightarrow$ $P(x,\vec{y})$ semi-decidable

EXERCISE: Define a function total and not computable $f: \mathbb{N} \to \mathbb{N}$ such that $f(x) = x$ on infinitely many $x \in \mathbb{N}$

($\{x \mid f(x) = x\}$ infinite)

1st solution



$$f(x) = \begin{cases} x & \text{if } x \text{ odd} \\ \varphi_{x/2}(x)+1 & \text{if } x \text{ even } \varphi_{x/2}(x)\downarrow \\ 0 & \text{if } x \text{ even } \varphi_{x/2}(x)\uparrow \end{cases}$$

→ f total

→ $\{x \mid f(x) = x\} \supseteq$ odd numbers infinite

→ f not computable since it differs from all total computable functions

($\forall x$ if φ_x is total $f(2x) = \varphi_x(2x)+1 \neq \varphi_x(2x)$)

$\rightsquigarrow$ $\varphi_x \neq f$

2nd solution

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_x(x)\downarrow \\ x & \text{if } \varphi_x(x)\uparrow \end{cases}$$

→ f is total

→ f is not computable ($\forall x$. if φ_x is total then $\varphi_x \neq f$

$\varphi_x(x) \neq \varphi_x(x)+1 = f(x)$)

→ $\{x \mid f(x) = x\} \supseteq \overline{K}$ infinite (e.g. because it is not recursive)

3rd solution

$$f(x) = \begin{cases} x+1 & \text{if } \varphi_x(x)\downarrow \\ x & \text{otherwise} \end{cases}$$

- f total
- $\{x \mid f(x) = x\} \supseteq \overline{K}$ infinite (e.g. because it is not recursive)
- not computable

$\chi_K(x) = f(x) \dot{-} x$ (if f computable, K recursive)

EXERCISE

Let $f: \mathbb{N} \to \mathbb{N}$ be computable

and assume $g(x) = f(x)$ almost everywhere ($\{x \mid f(x) \neq g(x)\}$ finite)

Then g is computable.