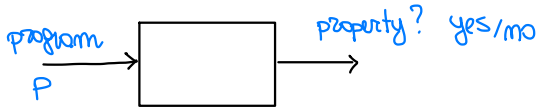
COMPUTABILITY (23/11/2026)

Rice's Theorem



> every property of programs which regards the behaviour is undecidable

"P is terminating on every input"
"P has some specific $n \in \mathbb{N}$ in output"
"P has dead code"
"P computes function"
.....

} undecidable

"the length of P is ≤ 10"
"program P is not using jumps"
.....

} decidable

What is a behavioural property of programs ?

$$A \subseteq \mathbb{N}$$

↑ set of programs (property)

$$T = \{ n \mid P_n = \gamma^{-1}(n) \text{ is terminating on every input} \}$$
$$= \{ n \mid \varphi_n \text{ is total} \}$$

$$ONE = \{ n \mid P_n \text{ is correctly implementing } \mathbb{1} \}$$
$$= \{ n \mid \varphi_n = \mathbb{1} \}$$

program property $A \subseteq \mathbb{N}$ is behavioural if for all $n \in \mathbb{N}$ (programs) the fact that $n \in A$ or $n \notin A$ only depends on φ_n

Def (extensional / saturated set) : $A \subseteq \mathbb{N}$ is **saturated (extensional)**

if $\forall m, n \in \mathbb{N}$

if $m \in A$ and $\varphi_m = \varphi_n$ then $n \in A$

$\Updownarrow$

$A = \{ n \in \mathbb{N} \mid \varphi_n \text{ satisfies some property} \}$

$= \{ n \in \mathbb{N} \mid \varphi_n \in \mathcal{A} \}$

for suitable $\mathcal{A} \subseteq \mathcal{Y}$

↑ property of functions

Examples

* $T = \{ n \mid P_n \text{ is terminating on every input} \}$

$= \{ n \mid \varphi_n \text{ is total} \}$

$= \{ n \mid \varphi_n \in \mathcal{T} \}$ $\quad \mathcal{T} = \{ f \in \mathcal{Y} \mid f \text{ is total} \}$

$\Updownarrow$

$\text{dom}(f) = \mathbb{N}$

* $ONE = \{ n \mid \varphi_n \text{ is a correct implementation of } \mathbb{1} \}$

$= \{ n \mid \varphi_n = \mathbb{1} \}$

$\Updownarrow$

$\varphi_n \in \{\mathbb{1}\}$

* $LEN\,10 = \{ n \mid P_n = \gamma^{-1}(n) \text{ has } \leq 10 \text{ lines of code} \}$

$n \in LEN\,10$

$n \notin LEN\,10$ s.t. $\varphi_n = \varphi_n$

e.g. $P = Z(1)$

$n = \gamma(P)$ $\quad \varphi_n(x) = 0 \quad \forall x$

$n \in LEN\,10$

P' $\begin{cases} Z(1) \\ Z(2) \\ \vdots \\ Z(2) \end{cases}$ 10 times

$n = \gamma(P')$ $\quad \varphi_n(x) = 0 \quad \forall x$

$n \notin LEN\,10$

$\varphi_n = \varphi_n$

LEN 10 not extensional

* NODEAD $= \{ m \mid P_m \text{ has no dead code} \}$

$m = \gamma(Z(1))$ $\qquad \varphi_m = 0 \qquad m \in \text{NODEAD}$

$$m = \gamma \begin{pmatrix} Z(1) \\ J(1,1,\text{END}) \\ S(1) \\ \text{END:} \end{pmatrix}$$

$\parallel$

$\varphi_m = 0 \qquad m \notin \text{NODEAD}$

NODEAD not saturated

* $K = \{ m \mid \varphi_m(m)\downarrow \}$

$= \{ m \mid \varphi_m \in \mathcal{K} \}$

$\mathcal{K} = \{ f \mid f(?)\downarrow \}$ apparently not extensional

to prove that K not saturated we need $m, n \in \mathbb{N}$

$$\begin{cases} m \in K & \varphi_m(m)\downarrow \\ n \notin K & \varphi_n(n)\uparrow \end{cases} \quad \text{and} \quad \varphi_m = \varphi_n$$

assume that we can show the existence of some program $m \in \mathbb{N}$

$$\varphi_m(x) = \begin{cases} 0 & \text{if } x = m \\ \uparrow & \text{otherwise} \end{cases} \qquad (*)$$

we have

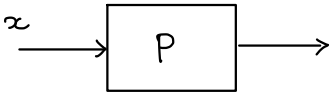
① $m \in K$ $\qquad \varphi_m(m) = 0$ $\qquad$ i.e. $\varphi_m(m)\downarrow$

② since there are infinitely many programs for φ_m, there is $n \neq m$ s.t. $\varphi_n = \varphi_m$

③ $n \notin K$ $\qquad \varphi_n(n) = \varphi_m(m)\uparrow$

K not saturated under assumption (*)

Can we build a program satisfying (*) ?



0 if $x = P$

$\uparrow$ if $x \neq P$

```
def P(x):
    if x = " def P(x):
            if x = \" --
                   "
```

? (cf. Quine program)

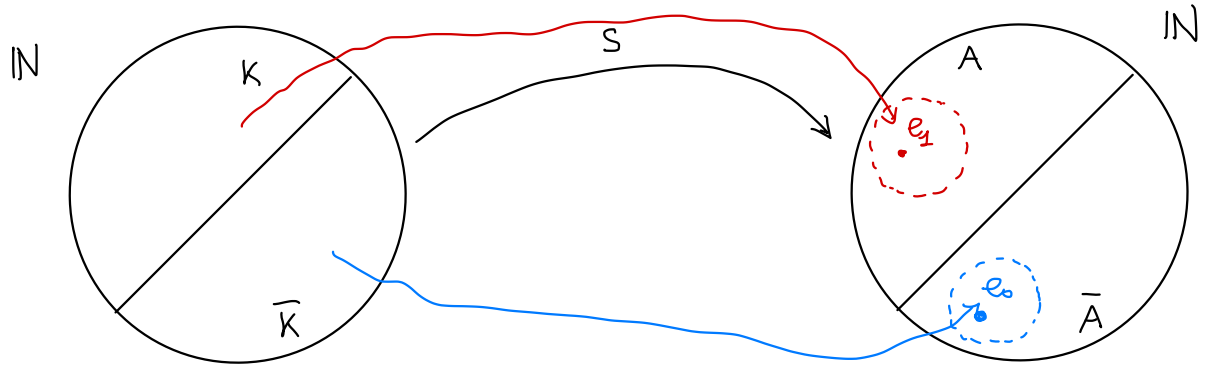
RICE'S THEOREM

Let $A \subseteq \mathbb{N}$ if A is saturated and $A \neq \emptyset$, $A \neq \mathbb{N}$

then A is not recursive

proof

we prove $K \leq_m A$ (since K is not recursive this implies A not recursive)



Let $e_0 \in \mathbb{N}$ be s.t. $\varphi_{e_0}(x) \uparrow \ \forall x$ (e_0 is a program for the function always undefined)

① Assume $e_0 \in \overline{A}$

take $e_1 \in A$ (it exists because $A \neq \emptyset$)

define

$$g(x,y) = \begin{cases} \varphi_{e_1}(y) & \text{if } x \in K \\ \varphi_{e_0}(y) & \text{if } x \notin K \end{cases}$$

$$= \begin{cases} \varphi_{e_1}(y) & \text{if } x \in K \\ \uparrow & \text{if } x \notin K \end{cases}$$

$$= \mathbb{1}\left(\varphi_x(x)\right) \cdot \varphi_{e_1}(y)$$

if $x \in K$: $\varphi_x(x)$ converges, so $\mathbb{1}(\varphi_x(x)) = 1$, and $1 \cdot \varphi_{e_1}(y) = \varphi_{e_1}(y)$

if $x \notin K$: $\varphi_x(x)$ diverges, so $\mathbb{1}(\varphi_x(x))$ is undefined, and the whole expression is undefined

$$= \mathbb{1}\left(\psi_U(x,x)\right) \cdot \varphi_{e_1}(y)$$ computable

By smn theorem there is a total computable function $s : \mathbb{N} \to \mathbb{N}$ s.t. $\forall x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} \varphi_{e_1}(y) & \text{if } x \in K \\ \varphi_{e_0}(y) & \text{if } x \notin K \end{cases}$$

I claim that s is the reduction function for $K \leq_m A$

* if $x \in K$ then $s(x) \in A$

let $x \in K$. Then $\varphi_{s(x)}(y) = \varphi_{e_1}(y)$ $\forall y \in \mathbb{N}$, i.e. $\varphi_{s(x)} = \varphi_{e_1}$.

Then since $e_1 \in A$ and A saturated, $s(x) \in A$.

* if $x \notin K$ then $s(x) \notin A$

let $x \notin K$. Then $\varphi_{s(x)}(y) = \varphi_{e_0}(y)$ $\forall y \in \mathbb{N}$, i.e. $\varphi_{s(x)} = \varphi_{e_0}$

since $e_0 \notin A$ and A saturated $s(x) \notin A$.

Hence s is indeed a reduction function for $K \leq_m A$, and since K is not recursive, neither A is.

(2) Assume $e_0 \in A$

if we let $B = \overline{A}$

then

$e_0 \notin B$

$B \neq \emptyset$ (since $A \neq \mathbb{N}$)

$B \neq \mathbb{N}$ (" $A \neq \emptyset$)

B is saturated (since A is so)

$\Rightarrow$ from (1)

B is not recursive

$\parallel$

$\overline{A}$

since $B = \overline{A}$ is not recursive, A not recursive

□

* Output problem

for some fixed $m \in \mathbb{N}$

$$B_m = \{ x \in \mathbb{N} \mid \varphi_x(y) = m \text{ for some } y \in \mathbb{N} \}$$

→ B_m is saturated

$$B_m = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{B}_m \}$$

$$\mathcal{B}_m = \{ f \in \mathcal{Y} \mid f(y) = m \text{ for some } y \in \mathbb{N} \}$$

→ $B_m \neq \emptyset$

e.g. let $e_1 \in \mathbb{N}$ be s.t. $\varphi_{e_1}(x) = x \ \forall x \in \mathbb{N}$ then $e_1 \in B_m$

since $\varphi_{e_1}(m) = m$

→ $B_m \neq \mathbb{N}$

e.g. let $e_0 \in \mathbb{N}$ be s.t. $\varphi_{e_0}(x) \uparrow \ \forall x \in \mathbb{N}$ then $m \notin E_{e_0} = \emptyset$

thus $e_0 \notin B_m$

Hence by Rice's theorem, B_m is not recursive.

EXERCISE :

$I = \{ x \mid P_x$ has infinitely many outputs $\}$

$= \{ x \mid E_x$ infinite $\}$ not recursive

* I is extensional

$I = \{ x \mid \varphi_x \in \mathcal{Y} \}$

where $\mathcal{Y} = \{ f \mid cod(f) \text{ infinite} \}$

* $I \neq \emptyset$

if e_1 is as above (index for identity) $E_{e_1} = \mathbb{N}$ infinite hence $e_1 \in I$

* $I \neq \mathbb{N}$

if e_0 is as above (index for function always undefined) $E_{e_0} = \emptyset$

hence $e_0 \notin I$

Hence, by Rice's theorem I is not recursive.

EXERCISE :

$A = \{ x \mid x \in W_x \cap E_x \}$

is it saturated ?

$A = \{ x \mid \varphi_x \in \mathcal{A} \}$

where $\mathcal{A} = \{ f \mid ? \in dom(f) \cap cod(f) \}$

apparently not saturated

use reduction

$K \leq_m A$

find a reduction function $s : \mathbb{N} \to \mathbb{N}$ total computable s.t.

$\forall x$

$x \in K$ iff $s(x) \in A$

$\Updownarrow$ $s(x) \in W_{s(x)} \iff \varphi_{s(x)}(s(x)) \downarrow$

and

$s(x) \in E_{s(x)} \iff \exists y \quad \varphi_{s(x)}(y) = s(x)$

↑ not necessarily $s(x)$

we define

$$g(x, y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{if } x \notin K \end{cases}$$

$\vdots$

$\varphi_{s(x)}$

$$= y \cdot \mathbb{1}(\varphi_x(x)) = y \cdot \mathbb{1}(\psi_U(x, x)) \quad \text{computable}$$

By smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total computable s.t.

$\forall x, y$

$$\varphi_{s(x)}(y) = g(x, y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

s is the reduction function for $K \leq_m A$

→ if $x \in K$ then $\varphi_{s(x)}(y) = y \quad \forall y \Rightarrow s(x) \in \underset{\mathbb{N}}{\underset{\shortparallel}{W_{s(x)}}} \cap \underset{\mathbb{N}}{\underset{\shortparallel}{E_{s(x)}}} = \mathbb{N}$

and thus $s(x) \in A$.

→ if $x \notin K$ then $\varphi_{s(x)}(y) \uparrow \quad \forall y \Rightarrow s(x) \notin \underset{\emptyset}{\underset{\shortparallel}{W_{s(x)}}} \cap \underset{\emptyset}{\underset{\shortparallel}{E_{s(x)}}} = \emptyset$

and thus $s(x) \notin A$

Hence $K \leq_m A$ and therefore A not recursive $\square$