COMPUTABILITY (10/11/2026)

OBSERVATION: A function which is total and not computable

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \psi_U(x,x) + 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

HALTING PROBLEM : The predicate below is UNDECIDABLE

$$Halt(x) = \begin{cases} \text{true} & \text{if } \varphi_x(x)\downarrow \quad (\text{i.e. } x \in W_x) \\ \text{false} & \text{if } \varphi_x(x)\uparrow \quad (\text{i.e. } x \notin W_x) \end{cases}$$

idea : by contradiction : we show that assuming Halt(x) decidable we can prove f computable

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \overbrace{\varphi_x(x)\downarrow}^{Halt(x)} \\ 0 & \text{otherwise} \end{cases}$$

$\not=$ $(\psi_U(x,x) + 1) \cdot \underbrace{\chi_{Halt}(x)}_{\substack{0 \text{ if } \varphi_x(x)\uparrow \\ 1 \text{ if } \varphi_x(x)\downarrow}}$

$\varphi_x(x)+1$ if $\varphi_x(x)\downarrow$; $\uparrow$ if $\varphi_x(x)\uparrow$

instead

$$f(x) = \text{“}\left(\mu(t,y,z).\left[(S(x,x,y,t) \wedge z = y+1 \wedge Halt(x)) \vee (z = 0 \wedge \neg Halt(x))\right]_z\right)\text{”}$$

“ $\rightsquigarrow y+1$ ” “ $\rightsquigarrow 0$ ”

$$= \left(\mu w.\ \left[S(x,x,(w)_2,(w)_1) \wedge (w)_3 = (w)_2 + 1 \wedge Halt(x)\right) \vee \left((w)_3 = 0 \wedge \neg Halt(x)\right)\right]_3$$

$(w)_1 = t$
$(w)_2 = y$
$(w)_3 = z$

if you call

$$Q(x,\omega) = \Big(S(x, x, (\omega)_2, (\omega)_1) \wedge (\omega)_3 = (\omega)_2 + 1 \wedge \text{Halt}(x) \Big) \vee$$
$$(\omega)_3 = 0 \wedge \neg \text{Halt}(x) \Big)$$

decidable

$$f(x) = \Big(\mu\omega . \; |\chi_Q(x,\omega) - 1| \Big)_3$$

computable (minimalisation / composition of computable functions)

$\Rightarrow$ contradiction

$\lightning$

Halt (x) not decidable. $\square$

EXERCISE : Let $Q(x)$ decidable predicate

$f_1, f_2 : \mathbb{N} \to \mathbb{N}$ computable

and define

$$f(x) = \begin{cases} f_1(x) & \text{if } Q(x) \\ f_2(x) & \text{if } \neg Q(x) \end{cases}$$

is computable

Proof

If f_1, f_2 are total

$$f(x) = f_1(x) \cdot \underbrace{\chi_Q(x)}_{\substack{= 1 \text{ if } Q(x) \text{ true} \\ 0 \text{ otherwise}}} + f_2(x) \cdot \underbrace{\chi_{\neg Q}(x)}_{\substack{= 1 \text{ if } Q(x) \text{ false} \\ 0 \text{ otherwise}}}$$

$f_1(x)$ if $Q(x)$ true, 0 otherwise

$f_2(x)$ if $Q(x)$ false, 0 otherwise

$\rightsquigarrow$ f is a composition of computable functions hence computable

In general, let $e_1, e_2 \in \mathbb{N}$ s.t. $\varphi_{e_1} = f_1$ and $\varphi_{e_2} = f_2$

$$f(x) = \Big(\mu(t,y) . \; \big(S(e_1, x, y, t) \wedge Q(x) \big) \vee \big(S(e_2, x, y, t) \wedge \neg Q(x) \big) \Big)_y$$

$$= \Big(\mu w. \quad \big(S(e_1, x_1, (w)_1, (w)_2) \wedge Q(x) \big) \vee \big(S(e_2, x_1, (w)_1, (w)_2) \wedge \neg Q(x) \big) \Big)_1$$

$(w)_1 = y$, $(w)_2 = t$

decidable

$\rightsquigarrow$ f is computable $\square$

EXERCISE : TOTALITY

$Tot(x) \equiv$ "φ_x is total" $\equiv$ "P_x is terminating on every input"

is undecidable.

Im fact

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } Tot(x) \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ f is total $\begin{cases} \text{if } Tot(x) & f(x) = \varphi_x(x) + 1 \downarrow \\ \text{if } \neg Tot(x) & f(x) = 0 \end{cases}$ always defined

$\rightarrow$ f is different from all total computable functions

(in fact $\forall x \in \mathbb{N}$ if φ_x is total then $f \neq \varphi_x$ on x

$$f(x) = \varphi_x(x) + 1 \neq \varphi_x(x)$$)

$\hookrightarrow$ f is not computable

Assume, by contradiction, that $Tot(x)$ is decidable. Then

$$f(x) = \begin{cases} f_1(x) & \text{if } Tot(x) \\ f_2(x) & \text{otherwise} \end{cases}$$

where

$f_1(x) = \varphi_x(x) + 1 = \psi_U(x, x) + 1$

$f_2(x) = 0$

computable functions

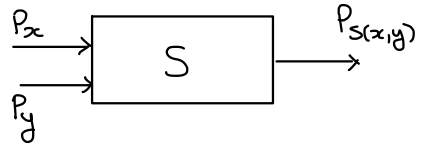
the f is computable (by previous exercise), contradiction!

$\Rightarrow$ $Tot(x)$ not decidable. $\square$

EFFECTIVE OPERATIONS ON COMPUTABLE FUNCTIONS

① there is a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$ such that

$$\forall x,y \qquad \varphi_{s(x,y)}(z) = \varphi_x(z) * \varphi_y(z) \qquad \forall z \in \mathbb{N}$$



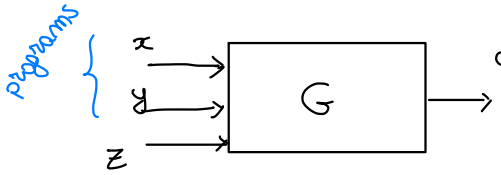
$P_x, P_y \rightsquigarrow$

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def P_s(x,y)(z):
    ν_x = P_x(z)
    ν_y = P_y(z)
    return ν_x * ν_y
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define a function $g : \mathbb{N}^3 \to \mathbb{N}$

$$g(x,y,z) = \varphi_x(z) * \varphi_y(z)$$
$$= \psi_U(x,z) * \psi_U(y,z)$$

computable (composition of computable functions)



function s takes G and hard code x, y into G

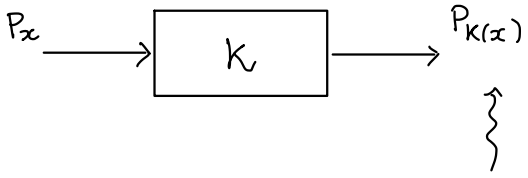
by smn theorem (corollary of) there is $s : \mathbb{N}^2 \to \mathbb{N}$ total computable such that $\forall x,y,z$

$$\varphi_{s(x,y)}(z) = g(x,y,z) = \varphi_x(z) * \varphi_y(z)$$

EXERCISE : Effectiveness of inverting a function

There is a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that

$\forall x \in \mathbb{N}$ if φ_x is injective then $\varphi_{k(x)}(z) = \varphi_x^{-1}(z) \quad \forall z$



computes the inverse of the function computed by P_x

idea : in order to compute $\varphi_x^{-1}(z)$ $\quad y=0\,?\quad P_x(0)=z\,?$ $\quad y=1\,?\quad P_x(1)=z\,?$

$\Leftsquigarrow$

define

$$g : \mathbb{N}^2 \to \mathbb{N}$$

$$g(x,z) = \varphi_x^{-1}(z) = \begin{cases} y & \text{s.t. } \varphi_x(y)=z \text{ when it exists} \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \left(\mu(y,t) \quad S(x,y,z,t) \right)_y$$

$$= \left(\mu w. \quad S(x,(w)_1,z,(w)_2) \right)_1$$

$$= \left(\mu w \mid \chi_S(x,(w)_1,z,(w)_2) - 1 \mid \right)_1$$

computable

Hence, by smn theorem, there is $k : \mathbb{N} \to \mathbb{N}$ total computable s.t.

$$\varphi_{k(x)}(z) = g(x,z) = (\varphi_x)^{-1}(z) \qquad \text{if } \varphi_x \text{ is injective}$$

What do we get when φ_x is not injective?

$\varphi_{k(x)}(z)$ is one of the (possibly many) y s.t. $\varphi_x(y)=z$

QUESTION: Given $f : \mathbb{N} \to \mathbb{N}$ computable, define for all $z \in \mathbb{N}$

$$f^{-1}[z] = \{ y \in \mathbb{N} \mid f(y) = z \}$$

and then let

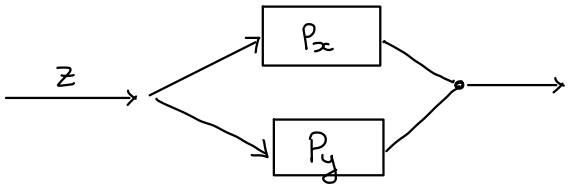
$$g(z) = \begin{cases} \min\left(f^{-1}[z]\right) & \text{if } f^{-1}[z] \neq \emptyset \\ \uparrow & \text{otherwise} \end{cases}$$

Is g computable?

EXERCISE : There is a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$ s.t.

$\forall x, y \in \mathbb{N}$ $\quad W_{s(x,y)} = W_x \cup W_y$

$\varphi_{s(x,y)}(z)\downarrow$ iff $\varphi_x(z)\downarrow$ or $\varphi_y(z)\downarrow$



$g : \mathbb{N}^3 \to \mathbb{N}$

$$g(x, y, z) = \begin{cases} \downarrow \text{ (1)} & \text{if } \varphi_x(z)\downarrow \text{ or } \varphi_y(z)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

$$\mathbb{1}\left(\mu t.\ H(x, z, t) \vee H(y, z, t)\right)$$

where $\mathbb{1} : \mathbb{N} \to \mathbb{N}$ $\quad \mathbb{1}(x) = 1 \ \ \forall x$

g is computable, hence by smn theorem there is $s : \mathbb{N}^2 \to \mathbb{N}$ total and computable s.t.

$$\varphi_{s(x,y)}(z) = g(x, y, z) = \begin{cases} 1 & \text{if } \varphi_x(z)\downarrow \text{ or } \varphi_y(z)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Hence s is the desired function, i.e. $W_{s(x,y)} = W_x \cup W_y$

$z \in W_{s(x,y)}$ iff $\varphi_{s(x,y)}(z)\downarrow$ iff $\varphi_x(z)\downarrow$ or $\varphi_y(z)\downarrow$

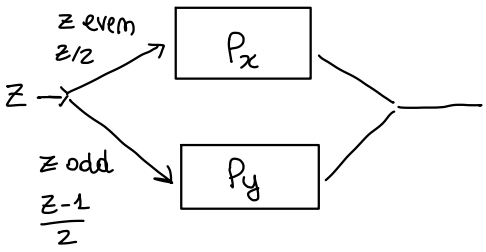
$\varphi_{s(x,y)} = g(x, y, z)$

iff $z \in W_x$ or $z \in W_y$

EXERCISE There is a total computable function $s: \mathbb{N}^2 \to \mathbb{N}$ such that

$$E_{s(x,y)} = E_x \cup E_y$$

($P_{s(x,y)}$ produces as outputs all and only the outputs of P_x and P_y)



	0	1	2	3	4
P_x	1	2	↑	4	0
P_y	↑	0	3	4	3

$$g(x,y,z) = \begin{cases} \varphi_x\left(z/2\right) & \text{if } z \text{ even} \\ \varphi_y\left(\frac{z-1}{2}\right) & \text{if } z \text{ odd} \end{cases}$$

~~$\varphi_v(x, qt(2,z)) \cdot \overline{sg}(rm(2,z)) +$~~
~~$\varphi_v(y, qt(2,z)) \cdot sg(rm(2,z))$~~

$$= \left(\mu(v,t).\ \left(S(x, z/2, v, t) \wedge z \text{ even}\right) \vee \left(S(y, \tfrac{z-1}{2}, v, t) \wedge z \text{ odd}\right)\right)_v$$

$$= \underbrace{\left(\mu w.\ \left(S(x, qt(2,z), (w)_1, (w)_2) \wedge z \text{ even}\right) \vee \left(S(y, qt(2,z), (w)_1, (w)_2) \wedge z \text{ odd}\right)\right)_1}_{\text{computable}}$$

By smn theorem there is $s: \mathbb{N}^2 \to \mathbb{N}$ total and computable s.t.

$$\varphi_{s(x,y)}(z) = g(x,y,z) = \begin{cases} \varphi_x\left(z/2\right) & \text{if } z \text{ is even} \\ \varphi_y\left(\frac{z-1}{2}\right) & \text{otherwise } (z \text{ is odd}) \end{cases}$$

I claim that s is the desired function. $E_{s(x,y)} = E_x \cup E_y$

($\subseteq$) $E_{S(x,y)} \subseteq E_x \cup E_y$

Let $n \in E_{S(x,y)}$ i.e. $\exists z$ s.t. $\varphi_{S(x,y)}(z) = n$

then either

$$\left.\begin{array}{l} n = \varphi_x(z/2) \rightsquigarrow n \in E_x \\ \text{or } n = \varphi_y\left(\frac{z-1}{2}\right) \rightsquigarrow n \in E_y \end{array}\right\} \rightarrow n \in E_x \cup E_y$$

($\supseteq$) $E_{S(x,y)} \supseteq E_x \cup E_y$

let $n \in E_x \cup E_y$ then either

(a) $n \in E_x \rightsquigarrow \exists z$ s.t. $\varphi_x(z) = n$

$\rightsquigarrow \varphi_{S(x,y)}(2z) = \varphi_x(2z/2) = \varphi_x(z) = n$

$\rightsquigarrow n \in E_{S(x,y)}$

or (b) $n \in E_y$

same

Hence we are done. □

EXERCISE : variant of URM

URM^P

$Z(m)$

~~$S(m)$~~ $P(m)$ $r_m \leftarrow r_m - 1$

$T(m,n)$

$J(m,n,b)$

$\mathcal{C}^P$ URM^P computable functions vs $\mathcal{C}$

EXERCISE : Are there $f, g : \mathbb{N} \to \mathbb{N}$ functions s.t.

① f computable, g not computable $f \circ g$ computable ?

② f not computable, g not computable $f \circ g$ computable ?