SOBOLEV SPACES (more suitable in some contexts than distributions → they are Banach spaces)

Ref EVANS chapt 5.

$U \subseteq \mathbb{R}^n$ $p \in [1, +\infty]$

$W^{1,p}(U) = \{ f \in L^p(U)$ f admits weak derivatives $\frac{\partial f}{\partial x_i}$ $\forall i$, $\frac{\partial f}{\partial x_i} \in L^p(U) \}$

$W^{2,p}(U) = \{ f \in L^p(U)$, f admits weak derivatives $\frac{\partial f}{\partial x_i} \in W^{1,p}(U) \}$

$k \in \mathbb{N}$

$W^{k,p}(U) = \{ f \in L^p(U)$ $\forall \alpha \in \mathbb{N}^n$ $|\alpha| \leq k$ $\exists v_\alpha \in L^p(U)$ such that $T_{v_\alpha} = D^\alpha T_f \}$

$\forall \phi \in C_c^\infty(U)$ $D^\alpha T_f(\phi) = \int_U v_\alpha \phi \, dx = (-1)^{|\alpha|} \int_U f D^\alpha \phi \, dx$

$\|f\|_{W^{k,p}} = \sum_{|\alpha|\le k} \|D^\alpha f\|_p$ (eq. $\left(\sum_{|\alpha|\le k} \|D^\alpha f\|_p^p\right)^{1/p}$)

Prop $(W^{k,p}(U), \|\cdot\|_{W^{k,p}})$ is BANACH SPACE

→ $p \in [1, +\infty)$ $W^{k,p}(U)$ is separable

→ $p \in (1, +\infty)$ $W^{k,p}(U)$ is reflexive

→ $p = 2$ $W^{k,2}(U)$ is Hilbert

$W^{k,2}(U) = H^k(U)$

$$(f,g)_{W^{1,2}(U)} = \int_U f \cdot g \, dx + \sum_{i=1}^{n} \int_U \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \, dx$$

proof $k = 1$ $W^{1,p}(U) \subseteq L^p(U) \times \underbrace{[L^p(U) \times \dots \times L^p(U)]}_{n \text{ time}} = (L^p(U))^{n+1}$

$$f \longmapsto \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

Continuous embedding

we claim $W^{1,p}(U)$ is closed in $(L^p(U))^{n+1}$ (the image of $W^{1,p}(U)$ is closed in $(L^p(U))^{n+1}$)

Let $f_k \in W^{1,p}$ and Cauchy

$\downarrow$

f_k is Cauchy in $L^p(U)$ $\quad$ $f_k \to f$ in $L^p(U)$

$\frac{\partial}{\partial x_i} f_k$ is Cauchy in $L^p(U)$ $\quad$ $\frac{\partial}{\partial x_i} f_k \to g_i$ in $L^p(U)$

Moreover $\forall \phi \in C_c^\infty(U)$ $\quad$ $\int_U \phi \frac{\partial}{\partial x_i} f_k \, dx = -\int_U \frac{\partial}{\partial x_i} \phi \, f_k \, dx$

$\downarrow$ $\qquad$ $\downarrow$

$\int_U \phi \, g_i \, dx = -\int_U \frac{\partial}{\partial x_i} \phi \, f \, dx$

$g_i = \frac{\partial}{\partial x_i} f$

$\Rightarrow f \in W^{1,p}(U)$.

So $W^{1,p}$ is closed $\Rightarrow W^{1,p}(U)$ is Banach space and

$1 < p < +\infty$ $\left(L^p(U)\right)^{n+1}$ is reflexive $\Rightarrow$ $W^{1,p}(U)$ is closed subspace of a reflexive space $\Rightarrow$ it is reflexive

$1 \leq p < +\infty$ $\left(L^p(U)\right)^{n+1}$ is separable $\Rightarrow$ $W^{1,p}(U)$ is a subspace of a separable space then it is separable.

$W^{1,2}(U)$ is Hilbert $\quad (f,g)_{W^{1,2}} = \int_U f \cdot g \, dx + \int_U \sum_i \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} dx$

Def $\quad W_0^{1,p}(U) = \overline{C_c^\infty(U)}^{\|\cdot\|_{1,p}} \qquad p \geq 1$

- $W_0^{1,p}(U) \subseteq W^{1,p}(U)$ (as a CLOSED SUBSPACE)

$(W_0^{1,p}(U), \|\cdot\|_{1,p})$ is a Banach space

Obs $W^{1,p}(U) \subsetneq L^p(U)$

in general new continuous functions for $p \leq n$.

U bdd set containing 0.

ex $f(x) = \frac{1}{|x|^\alpha} \in L^p(U) \iff p\alpha < n$

$f(x) \in W^{1,p}(U) \iff p \cdot (\alpha+1) < n$ $\Rightarrow$ in particular $p < n$!

$\frac{1}{|x|^\alpha} \in W^{1,p}(B(0,R)) \iff \alpha < \frac{n-p}{p}$ $(p<n)$

ex $f(x) = \lg\left(\lg\left(1+\frac{1}{|x|}\right)\right) \in L^p(U)$ $\forall p \in [1, +\infty)$

U bdd containing 0.

$\nabla f(x) = \frac{1}{\lg(1+\frac{1}{|x|})} \cdot \frac{1}{1+\frac{1}{|x|}} \cdot \left(-\frac{1}{|x|^2}\right) \frac{x}{|x|}$ is also the WEAK GRADIENT

$$|\nabla f(x)| = \frac{1}{\lg\left(\frac{|x|+1}{|x|}\right)} \cdot \frac{1}{|x|(|x|+1)} \approx \frac{1}{(-\lg|x|)} \frac{1}{|x|} \quad \text{for } x \to 0$$

$$\left(-\frac{1}{\lg|x|}\right) \frac{1}{|x|} \in L^n\left(B(0,\tfrac{1}{2})\right) \quad \text{in } \mathbb{R}^n \quad \text{for } n>1.$$

$$\int_{B(0,\frac{1}{2})} \left(\frac{-1}{|x|} \frac{1}{\lg|x|}\right)^n dx = \int_0^{1/2} \frac{1}{r^n} \frac{1}{|\lg r|^n} \cdot r^{n-1} n\omega_n \, dr =$$

$$= \int_0^{1/2} \frac{1}{r} \frac{1}{|\lg r|^n} n\omega_n \, dr = \int_{-\infty}^{\lg(\frac{1}{2})} n\omega_n \frac{1}{|s|^n} ds = \int_{\lg 2}^{+\infty} n\omega_n \frac{1}{s^n} ds < +\infty$$

$n>1$

So for $n>1$ $\quad \lg\left(\lg\left(1+\frac{1}{|x|}\right)\right) \in W^{1,n}(B(0,R)) \quad (p=n)$

for $p>n$ all the functions in $W^{1,p}(U)$ are CONTINUOUS.

for $p \geq 1$ $n=1$ all the functions in $W^{1,p}(I)$ are continuous

Characterization of Sobolev spaces in dim 1

Def (ABSOLUTELY CONTINUOUS FUNCTIONS)

$f:(a,b)\to\mathbb{R}$ is ABSOLUTELY CONTINUOUS ($f\in AC(a,b)$) if f is continuous, f' exists a.e, $f'\in L^1(a,b)$ and moreover $f(y)=f(x)+\int_x^y f'(t)\,dt$ for a.e $x,y\in(a,b)$

Obs If $f\in AC(a,b)$ then f' is also the weak derivative of f.

$$\int_a^b \phi'(x)f(x)\,dx=\int_a^b \phi'(x)\left[f(a)+\int_a^x f'(t)\,dt\right]dx=f(a)\int_a^b \phi'(x)\,dx+$$

($\phi(a)=0=\phi(b)$)

$$+\int_a^b\int_a^x \phi'(x)f'(t)\,dt\,dx=\int_a^b\int_t^b \phi'(x)f'(t)\,dx\,dt=\int_a^b(-\phi(t))f'(t)\,dt=$$

$$=-\int_a^b \phi(t)f'(t)\,dt \quad\Rightarrow\quad (T_f)'=T_{f'}.$$

Theorem Let $(a,b) \subseteq \mathbb{R}$ bold.

1) Then $W^{1,1}(a,b) = AC(a,b)$ $\quad f \in W^{1,1}(a,b) \leftrightarrow \exists \tilde{f}$ $\tilde{f} = f$ a.e. $\tilde{f} \in AC(a,b)$

2) $\forall f$ $f \in W^{1,p}(a,b)$ $p > 1$ $\quad (p \in (1, +\infty])$

Then $\exists \tilde{f} = f$ a.e. $\tilde{f} \in AC(a,b)$, $\tilde{f} \in C^{0,1-\frac{1}{p}}(a,b)$

$p = +\infty$ $\tilde{f} \in C^{0,1}(a,b)$ (Lipschitz)

$W^{1,p}(a,b) \subseteq AC(a,b) \cap C^{0,1-\frac{1}{p}}(a,b)$

3) $\|f\|_\infty \leq C \|f\|_{W^{1,p}}$ $\forall p \in [1, +\infty]$ $\exists C > 0$ such that and $\|f\|_{C^{0,1-\frac{1}{p}}} \leq C \|f\|_{W^{1,p}}$

4) Finally $f \in W^{1,p}(\mathbb{R}) \Rightarrow f \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, f absolutely continuous. (moreover $f \in C^{0,1-\frac{1}{p}}(a,b)$ $\forall (a,b) \subset\subset \mathbb{R}$)

PROOF $f \in AC(a,b) \rightarrow f \in W^{1,1}(a,b)$ by definition

$f \in W^{1,p}(a,b) \Rightarrow v$ weak derivative $\in L^p(a,b)$

$p \in [1, +\infty]$ $\int f \phi' dx = -\int_a^b v \phi \, dx$ $\forall \phi \in C_c^\infty(a,b)$