Definition $T \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution of $\mathcal{Y}$ differ. operator if $\mathcal{Y}(T) = \delta_0$.

Proposition If T is a differential solution to $\mathcal{Y}$ then $\forall \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ $T * \varphi \in \mathcal{C}^\infty(\mathbb{R})$ is a solution to $\mathcal{Y}(T * \varphi) = \varphi$

Observation $\forall \mathcal{Y}$ $\mathcal{Y}(T * \varphi) = \mathcal{Y}(T) * \varphi = T * \mathcal{Y}(\varphi)$.

$\mathcal{Y}(T * \varphi) = T * \mathcal{Y}(\varphi)$ by linearity and definition

$$\left(\frac{\partial}{\partial y_i} T\right) * \varphi(x) = \left(\frac{\partial}{\partial y_i} T\right)(\varphi^x) = -T\left(\frac{\partial}{\partial y_i} \varphi^x(y)\right) = T\left(\frac{\partial}{\partial x_i} \varphi^x\right) = \frac{\partial}{\partial x_i}(T * \varphi)(x)$$

proof of proposition

$$\mathcal{Y}(T * \varphi) = \{\mathcal{Y}(T) * \varphi\} = \delta_0 * \varphi = \varphi$$

since T is a FUNDAM. SOLUTION

$\uparrow$ SINCE δ_0 is the UNIT

Ex

Fundamental solution Laplacien

T_f where $f(x) = -\frac{1}{2\pi} \log|x|$ $n=2$

$= -\frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}$ $n>2$

Heat operator T_f $f = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$

Wave equation in dim $n=2$ $\partial_{tt} - \partial_{xx}$

$T = \frac{1}{2}\theta(t)\left[\delta_{x-t} + \delta_{x+t}\right]$ $\theta(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$

in dim $n=4$ $\partial_{tt} - \Delta_x$ $x \in \mathbb{R}^3$

$T = \frac{1}{4\pi t}\theta(t)\,\delta_{t-|x|}$

Obs Let $f = \frac{1}{-n(n-2)\omega_n} \frac{1}{|x|^{n-2}}$ $n>2$ or $f = -\frac{1}{2\pi} \log|x|$ $n=2$

So in particular if $\phi = C_c^\infty(\mathbb{R}^n)$ then

$$u(x) = T_f * \phi(x) = f * \phi(x) := -\frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^{n-2}} dy$$

satisfies $\Delta u(x) = \phi(x) \quad \forall x \in \mathbb{R}^n$

The operator $F : C_c^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$

$$\phi \longmapsto f * \phi(x) = -\frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^{n-2}} dy$$

is the "inverse" of the Laplacian

$$\Delta F(\phi) = \phi$$

F can be extended to $L^2(U)$ $U \subset \mathbb{R}^n$ bdd.

Def A differential operator $\mathcal{L}$ is **hypelliptic** if $\forall T \in \mathcal{D}'(\mathbb{R}^n)$ such that $\mathcal{L}(T) = 0$ there exists $f \in C^\infty(\mathbb{R}^n)$ with $T = T_f$.

Weyl Lemma Δ is hypelliptic.

(that is harmonic distributions are smooth, i.e
if $T(\Delta\phi) = 0 \; \forall \phi \in C_c^\infty(\mathbb{R}^n)$ $T = T_f$ for $f \in C^\infty(\mathbb{R}^n)$

Obs also the heat operator in $\mathbb{R}^{n+1} = (t, x)$ given by

$$\mathcal{L}(\phi) = \frac{\partial \phi}{\partial t} - \sum_1^n \frac{\partial^2 \phi}{\partial x_i^2} \qquad \left(\frac{\partial}{\partial t} - \Delta_x\right)$$

is hypelliptic

the wave operator in $\mathbb{R}^{n+1}$ is NOT hypelliptic

$$\mathcal{L}(\phi) = \frac{\partial^2 \phi}{\partial t^2} - \sum_1^n \frac{\partial^2 \phi}{\partial x_i^2} \qquad \left(\frac{\partial^2}{\partial t^2} - \Delta_x\right)$$

Lemma Let $f \in L^1_{loc}(\mathbb{R}^n)$ s.t. that for a.e. x $\forall r>0$ it holds (MV) $f(x) = \frac{1}{\omega_n r^n} \int_{B(x,r)} f(y)\,dy$.

Then (1) f satisfies (MV) for a.e. x $\forall r>0$ $f(x) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} f(y)\,dS$

(2) $f * \rho_\varepsilon = f \Rightarrow f \in C^\infty(\mathbb{R}^n)$

(3) $\Delta f = 0$

proof (1) Let x such that (MV) holds.

$$m(r) = \int_{B(x,r)} f(y)\,dy = \frac{1}{\omega_n r^n} \int_0^r \int_{\partial B(x,s)} f(y)\,d\mathcal{H}^{n-1}(y)\,ds \equiv f(x) \Rightarrow m'(r) = 0$$

$$m'(r) = -\frac{n}{\omega_n r^{n+1}} \underbrace{\int_{B(x,r)} f(y)\,dy}_{= f(x)\omega_n r^n} + \frac{1}{\omega_n r^n} \int_{\partial B(x,r)} f(y)\,d\mathcal{H}^{n-1}(y) = 0$$

$$-\frac{n}{r} f(x) + \frac{1}{\omega_n r^n} \int_{\partial B(x,s)} f(y)\,d\mathcal{H}^{n-1}(y) = 0 \Rightarrow (MV) \text{ holds}$$

Obs $(SMV) \Rightarrow (MV)$ take $r>0$

$$\int_{B(x,r)} f(y)\,dy = \int_0^r \int_{\partial B(x,s)} f(y)\,d\mathcal{H}^{n-1}(y)\,ds \overset{(SMV)}{=} \int_0^r f(x)\, n\omega_n s^{n-1} ds$$

$$= \omega_n r^n f(x) \qquad (MV) \text{ holds.}$$

(2)

Let $f * \rho_\varepsilon(x) = \int_{B(x,\varepsilon)} \rho_\varepsilon(x-y) f(y)\,dy = \qquad \rho_\varepsilon(x-y) = \bar{\rho}_\varepsilon(|x-y|)$

$$= \int_0^\varepsilon \bar{\rho}_\varepsilon(s) \int_{\partial B(x,s)} f(y)\,d\mathcal{H}^{n-1}(y)\,ds = \int_0^\varepsilon \bar{\rho}_\varepsilon(s) f(x)\, n\omega_n s^{n-1} ds =$$

$$= f(x) \cdot \int_0^\varepsilon \bar{\rho}_\varepsilon(s)\, n\omega_n s^{n-1} ds = f(x) \int_{B(0,\varepsilon)} \rho_\varepsilon(y)\,dy = f(x)$$

$\Rightarrow$ for a.e. x $\quad f * \rho_\varepsilon = f \Rightarrow f \in C^\infty(\mathbb{R}^n)$.

(3) By (1) $\quad \phi(r) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} f(y)\,d\mathcal{H}^{n-1} \equiv f(x) \qquad \phi'(r) = 0$

$$\sigma(r) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(0,1)} f(x+tr)\, r^{n-1}\, d\mathcal{H}^{n-1}(t)$$

$$\sigma'(r) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} \nabla f(x+tr)\cdot t \; d\mathcal{H}^{n-1}(t) =$$

$$= \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} \nabla f(y)\cdot\left(\frac{y-x}{r}\right) d\mathcal{H}^{n-1}(y) \qquad \frac{y-x}{r} = \nu_{B(x,r)}(y)$$

$$= \text{divergence thm} = \frac{1}{n\omega_n r^{n-1}} \int_{B(x,r)} \underbrace{\operatorname{div}(\nabla f(y))}_{\Delta f(y)}\, dy$$

$$\Rightarrow \forall x \in \mathbb{R}^n \;\forall r>0 \qquad \int_{B(x,r)} \Delta f(y)\,dy = 0 \Rightarrow \Delta f(x) = 0 \qquad \square$$

Observation:

Also true that $f \in C^\infty(\mathbb{R}^n)$ $\Delta f(x)=0 \Rightarrow f$ satisfies (SMV) $\Rightarrow f$ satisfies (MV)

[Since $\Delta f(x)=0 \Rightarrow \sigma'(r)=0 \;\forall r \;\forall x \Rightarrow$ (SMV) $\Rightarrow$ (MV)]

proof of WEYL LEMMA (sketch)

Let $\Delta T = 0$. ρ_ε a mollifier $\quad T * \rho_\varepsilon \in C^\infty(\mathbb{R}^n)$

$\Delta(T * \rho_\varepsilon) = \Delta T * \rho_\varepsilon = 0 \Rightarrow T * \rho_\varepsilon$ is harmonic $\Rightarrow$ (REMARK)

$T * \rho_\varepsilon$ satisfies (SMV) and (MV)

$$(T * \rho_\varepsilon) * \rho_\delta = T * \rho_\varepsilon$$

Remember property of convolution: $T(\varphi * \psi) = \int_{\mathbb{R}^n} T(\varphi^x)\, \psi(x)\,dx$ $\quad \varphi(x) = \varphi(-x)$

$$\varphi * \psi(y) = \int_{\mathbb{R}^n} \varphi(y-x)\,\psi(x)\,dx = \int_{\mathbb{R}^n} \varphi(x-y)\,\psi(x)\,dx =$$

$$= \int_{\mathbb{R}^n} \varphi^y(y)\,\psi(x)\,dx \qquad T(\varphi * \psi(y)) = \int_{\mathbb{R}^n} T(\varphi^x)\,\psi(x)\,dx$$

$$(T * \varphi_\varepsilon) * \varphi_\delta(x) = \int_{\mathbb{R}^n} (T * \varphi_\varepsilon)(y)\, \varphi_\delta(x-y)\, dy =$$

$$= \int_{\mathbb{R}^n} T(\varphi_\varepsilon^y)\, \varphi_\delta(x-y)\, dy = \int_{\mathbb{R}^n} T(\varphi_\varepsilon^y)\, \check{\varphi}_\delta^x(y)\, dy =$$

$$= T(\varphi_\varepsilon * \check{\varphi}_\delta^x) = T(\check{\varphi}_\delta^x * \varphi_\varepsilon) = (T * \varphi_\delta) * \varphi_\varepsilon(x)$$

$$\varphi_\delta^x * \varphi_\varepsilon(z) = \int_{\mathbb{R}^n} \varphi_\varepsilon(y)\, \varphi_\delta^x(z-y)\, dy =$$

$$= \int_{\mathbb{R}^n} \varphi_\varepsilon(y)\, \varphi_\delta(x-z+y)\, dy = \int_{\mathbb{R}^n} \varphi_\varepsilon(y)\, \varphi_\delta(x-z-y)\, dy =$$

$$= \left(w = x+y\right) = \int_{\mathbb{R}^n} \varphi_\varepsilon(w-x)\, \varphi_\delta(z-w)\, dw = \int_{\mathbb{R}^n} \varphi_\varepsilon(x-w)\, \varphi_\delta(w-z)\, dw$$

$$= \int_{\mathbb{R}^n} \check{\varphi}_\varepsilon^x(w)\, \varphi_\delta(w-z)\, dw = \varphi_\delta * \check{\varphi}_\varepsilon^x(z)$$

Now $(T * \varphi_\varepsilon) * \varphi_\delta(x) = T * \varphi_\varepsilon(x)$ since

$\Delta(T * \varphi_\varepsilon) = 0$

$$(T * \varphi_\varepsilon) * \varphi_\delta = T * \varphi_\varepsilon \xrightarrow{\varepsilon \to 0} T$$

$$\| \qquad\qquad\qquad\qquad \Big\downarrow \; T = T * \varphi_\delta \Rightarrow (T \in C^\infty)$$

$$(T * \varphi_\delta) * \varphi_\varepsilon \xrightarrow{\varepsilon \to 0} T * \varphi_\delta$$