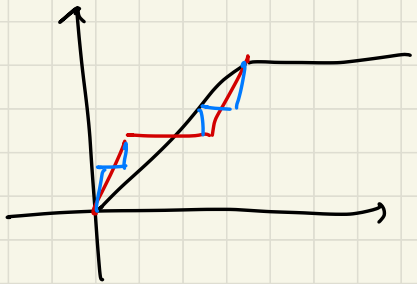
CANTOR VITALI FUNCTION

$$f_0(x) = \begin{cases} 0 & x \le 0 \\ x & x \in (0,1) \\ 1 & x \ge 1 \end{cases}$$

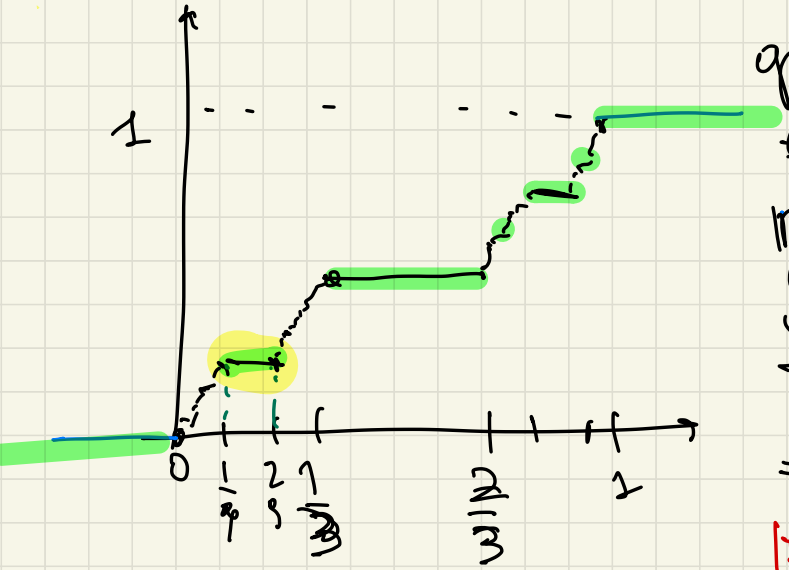


$$f_{n+1}(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{2} f_n(3x) & x \in (0, \frac{1}{3}) \\ \frac{1}{2} & x \in (\frac{1}{3}, \frac{2}{3}) \\ \frac{1}{2} + \frac{1}{2} f_n(3x-2) & x \in (\frac{2}{3}, 1) \\ 1 & x \ge 1 \end{cases}$$

$$\|f_{n+1} - f_n\|_\infty \le \frac{1}{3\,2^{n+1}} \Rightarrow f_n \to f \text{ uniformly}$$

$\Rightarrow f : \mathbb{R} \to [0,1]$ is continuous, monotone non decreasing, constant on $\mathbb{R} \setminus C$.

CANTOR-VITALI FUNCTION



graph of f is the DEVIL'S STAIRCASE

f is increasing only on C, $|C|=0$.

$\mathbb{R}\setminus C$ is union of disjoint intervals (on each of which f is constant).

$f:\mathbb{R}\to[0,1]$

$f|_C$ is injective ($\mathrm{card}\, C = \mathrm{card}\,[0,1]$)

$|f(x)-f(y)| \le |x-y|^{\log_3 2}$

$f'(x)=0$ for a.e. x (since it is constant on every interval I, $I\cap C=\emptyset$ and $|C|=0$)

Tf' is a positive distribution $\to$ so it has order 0 $\Rightarrow \exists\,\mu$ measure (Radon) $Tf'(\phi)=\int_{\mathbb{R}}\phi\,d\mu$

$\forall \phi \in C_c^\infty(\mathbb{R})$

μ is SUPPORTED on $C \Longrightarrow \mu \perp \mathcal{L}$ ($\mu \neq 0$!)

μ is the derivative in the sense of distributions of a continuous monotone function.

(so in particular $\mu\{x\} = 0 \quad \forall x \in \mathbb{R}$)

Def μ Radon measure on $\mathbb{R}^n$ is a **CANTORIAN MEASURE** if $\mu \perp \mathcal{L}^n$ (μ is singular with respect to Lebesgue) and μ has no jump part (that is $\mu\{x\} = 0 \quad \forall x \in \mathbb{R}^n$)

↓

in dim 1 : μ is a Cantorian measure if it is the distributional derivative of f, monotone non decreasing, continuous, and with $f'(x) = 0$ for a.e. $x \in \mathbb{R}$.

$f \in L^1_{loc}(\mathbb{R})$ T_f distribution of order 0.

Assume that also T_f' is a distribution of order 0.
that is a SIGNED RADON MEASURE!

$\Rightarrow \exists \mu^+, \mu^-$ Radon $T_f' = T_{\mu^+ - \mu^-}$

$\rightarrow \mu^+ = \mu^+_{ac} + \mu^+_s$ (exactly the same for μ^-)

μ^+_s has a jump part
and a CANTORIAN PART

$\mu^+_s = \mu^+_J + \mu^+_C$ $\quad \mu^+_J = \sum_{i \in I} c_i \delta_{a_i}$ I countable at most

μ^+_C Cantorian measure

$\rightarrow \mu^+_{ac} = \nu^+ dx$ for some $\nu^+ \in L^1_{loc}$ $\quad \nu^+ \geq 0$
$\mu^-_{ac} = \nu^- dx$ for some $\nu^- \in L^1_{loc}$ $\quad \nu^- \geq 0$

If $(T_f)' = T_{\mu^+ - \mu^-}$ where $\mu^+, \mu^- \ll \mathcal{L}$ then

$$(T_f)' = T_v \qquad \exists v \in L^1_{loc} \qquad \mu^+ = v^+ dx \quad \mu^- = v^- dx$$

(that is the distributional derivative of f is a measure which is absolutely continuous w.r.t. $\mathcal{L}$)

→ f is absolutely continuous

and $v \in L^1_{loc}$ such that $(T_f)' = T_v$

is called **the weak derivative of f**

(coincide with the a.e. derivative if it is defined)

(Note that it is possible that f DOES NOT ADMIT a.e. derivative but does weak derivative .. ex: $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

f' does not exist, weak derivative is 0.

More generally $U \subseteq \mathbb{R}^n$ open

Def Let $f \in L^1_{loc}(U)$ and $T_f(\phi) = \int_U \phi f \, dx \quad \forall \phi \in C_c^\infty(U)$

There f admits **WEAK DERIVATIVE** in the e_i-direct.

if $\exists v_i \in L^1_{loc}(U)$ s.t. $\frac{\partial}{\partial x_i} T_f = T_{v_i}$

that is $\forall \phi \in C_c^\infty(U)$ $\quad (-1) \int_U f(x) \frac{\partial}{\partial x_i} \phi \, dx = \int_U v_i \, \phi \, dx$.

WEAK GRADIENT $Df = (v_1, \dots, v_n)$

WEAK DERIVATIVE $\frac{\partial f}{\partial x_i} = v_i$

Prop If a weak derivative exists it is unique.

Assume not true v_i, w_i weak derivatives of f in the e_i-direction

$$\downarrow \quad \int_U v_i \phi \, dx = -\int_U f \frac{\partial \phi}{\partial x_i} dx = \int_U w_i \phi \, dx \Rightarrow \int_U (v_i - w_i) \phi \, dx = 0 \quad \forall \phi \in C_c^\infty(U)$$

$\Rightarrow$ LEMMA DI DEBOIS-REYMOND $v_i - w_i = 0$ a.e. $\Rightarrow v_i = w_i$ $\square$

Analogously if $f \in L^1_{loc}(\mathbb{R}^n)$
the weak α-derivative of f (if it exists) is

$v \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} v \cdot \phi \, dx \qquad \forall \phi \in C^\infty_c(\mathbb{R}^n)$$

($v = D^\alpha f$ in weak sense)

To remark

$f \in L^1_{loc}(\mathbb{R}^n)$ → $D^\alpha f$ in weak sense is a function in $L^1_{loc}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} f D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \beta^\alpha f \, \phi \, dx \quad \forall \phi \in C^\infty_c(\mathbb{R}^n)$

→ $D^\alpha T_f$ in the sense of DISTRIBUTION is a DISTRIBUTION (of order $\leq |\alpha|$) such that $D^\alpha T_f(\phi) = (-1)^{|\alpha|} T_f(D^\alpha \phi)$.

Example $f(x) = \frac{1}{|x|^\alpha}$ $x \in \mathbb{R}^n$ $\boxed{\alpha < n-1}$ $\boxed{f \in L^1_{loc} \iff \alpha < n}$

$\frac{\partial f}{\partial x_i}$ in the weak sense coincide with the e.e. derivative.

$\frac{\partial f}{\partial x_i} = -\alpha \frac{x_i}{|x|^{\alpha+2}}$ Observe that $\boxed{\frac{\partial f}{\partial x_i} \in L^1_{loc} \iff \alpha + 1 < n \quad (\alpha < n-1)}$

$\forall \phi \in C^\infty_c(\mathbb{R}^n)$ $\frac{\partial}{\partial x_i} T_f(\phi) = -T_f\left(\frac{\partial}{\partial x_i}\phi\right) =$

$$= -\int_{\mathbb{R}^n} f(x) \frac{\partial}{\partial x_i}\phi \, dx = -\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{1}{|x|^\alpha} \frac{\partial}{\partial x_i}\phi \, dx =$$

($f \uparrow L^1_{loc}$)

$$= (\text{Gauss Green}) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \phi \frac{\partial f}{\partial x_i} dx - \int_{\partial B(0,\varepsilon)} \phi \cdot f \cdot \underbrace{x_i}_{\downarrow} \, dS$$

exterior normal to $\mathbb{R}^n \setminus B(0,\varepsilon)$

$$= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \phi(x) \frac{\partial}{\partial x_i} f \, dx + \int_{\partial B(0,\varepsilon)} \phi(x) \frac{1}{\varepsilon^n} \cdot \frac{x_i}{\varepsilon} \, d\mathcal{H}^{n-1}(x)$$

$$\leq \|\phi\|_\infty \cdot \frac{\mathcal{H}^{n-1}(\partial B(0,\varepsilon))}{\varepsilon^n} = \|\phi\|_\infty \, n\omega_n \frac{\varepsilon^{n-1}}{\varepsilon^n}$$

$$\longrightarrow 0$$

$$= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \phi(x) \frac{\partial}{\partial x_i} f \, dx$$

$$= \int_{\mathbb{R}^n} \phi(x) \frac{\partial}{\partial x_i} f \, dx -$$

Def $\mathcal{Y}$ is a differential operator of order k with constant coefficients if

$\forall \phi \in \mathcal{C}^\infty \quad \mathcal{Y}(\phi) = \sum_{|\alpha| \le k} c_\alpha D^\alpha \phi \qquad c_\alpha \in \mathbb{R}$

ex $\mathcal{Y}(\phi) = \Delta\phi = \frac{\partial^2}{\partial x_1^2}\phi + \dots + \frac{\partial^2}{\partial x_n^2}\phi$

$\forall \phi \in \mathcal{C}^\infty$

$c_\alpha = 1 \iff \alpha = (2,0\dots0), \ \alpha = (0,2,\dots)$

otherwise $c_\alpha = 0$

If $T \in \mathcal{D}'(\mathbb{R}^n)$

then $\mathcal{Y}(T) = \sum_{|\alpha| \le k} c_\alpha D^\alpha T \Rightarrow \mathcal{Y}(T)(\varphi) = \sum_{|\alpha| \le k} c_\alpha (-1)^{|\alpha|} T(D^\alpha \phi)$

ex $\Delta(T) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} T \qquad \Delta T(\phi) = T(\Delta\phi)$

$(\partial_t - \Delta_x) T(\phi) = T\left(-\frac{\partial}{\partial t}\phi - \Delta_x \phi\right) \dots$

Def Let $\mathcal{J}$ be a differential operator of order k with constant coefficients.

$T \in \mathcal{D}'(\mathbb{R}^n)$ is a FUNDAMENTAL SOLUTION of $\mathcal{J}$ if $\mathcal{J}(T) = \delta_0$

(that is $\mathcal{J}(T)(\varphi) = \varphi(0) \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n)$).

FUNDAMENTAL SOLUTION of THE LAPLACIAN.

- On $\mathbb{R}^2$ $T = T_f$ $f = -\frac{1}{2\pi} \log|x|$ is the fundamental solution of the laplacian
- On $\mathbb{R}^n$, $n > 2$ $T = T_f$ $f = -\frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}$ is the fundamental solution of the laplacian.

proof $n > 2$ $x \neq 0$ $\nabla f = + \frac{1}{n\omega_n} \frac{x}{|x|^n}$

$$\Delta f = \operatorname{div}(\nabla f) = + \frac{1}{n\omega_n} \frac{n}{|x|^n} - \frac{1}{n\omega_n} n \frac{|x|^2}{|x|^{n+2}} = 0$$

$\forall \phi \in C_c^\infty(\mathbb{R}^n)$

$$\Delta T_f(\phi) = T_f(\Delta\phi) = \int_{\mathbb{R}^n} f \Delta\phi \, dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} f\Delta\phi - \Delta f \cdot \phi \, dx =$$

($\Delta f = 0$)

$$= \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} f \cdot \nabla\phi \cdot \frac{(-x)}{|x|} - \phi \nabla f \cdot \left(-\frac{x}{|x|}\right) d\mathcal{H}^{n-1}(x) =$$

↓ DIVERGENCE THM

$f\Delta\phi - (\Delta f)\phi = \operatorname{div}(f\nabla\phi - \nabla f \cdot \phi)$

$$\left| \int_{\partial B(0,\varepsilon)} f \cdot \nabla\phi\left(\frac{x}{|x|}\right) \, d\mathcal{H}^{n-1} \right| \leq \underbrace{\frac{1}{n\omega_n} \frac{1}{\varepsilon^{n-2}}}_{\varepsilon} \cdot \|\nabla\phi\|_\infty \cdot \mathcal{H}^{n-1}(\partial B(0,\varepsilon))$$

$$= \varepsilon \|\nabla\phi\|_\infty \longrightarrow 0$$

$$= \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} (-\phi(x)) \cdot \left(+\frac{1}{n\omega_n} \frac{x}{|x|^n}\right) \cdot \left(-\frac{x}{|x|}\right) d\mathcal{H}^{n-1} =$$

$$= \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} \phi(x) \frac{1}{n\omega_n \varepsilon^{n-1}} \, d\mathcal{H}^{n-1} = \phi(0). \quad \square$$