Differential operators on distributions

Let $T \in \mathcal{D}'(U)$ $\quad \alpha = (\alpha_1, \dots, \alpha_n)$ $\quad \alpha_i \in \mathbb{N}$

$D^\alpha T$ is the distribution defined as

$$D^\alpha T(\varphi) := (-1)^{|\alpha|} T(D^\alpha \phi) \qquad \forall \phi \in \mathcal{C}_c^\infty(U).$$

Note that since $\phi \in \mathcal{C}_c^\infty(U) \rightarrow \partial_{x_i}^{\alpha_i} \partial_{x_j}^{\alpha_j} \phi = \partial_{x_j}^{\alpha_j} \partial_{x_i}^{\alpha_i} \phi$

The definition is a generalization of Gauss-Green / integration by parts

If T is associated to $f \in \mathcal{C}^1(U) \cap L^1_{loc}(U)$

$$T = T_f \rightarrow \left(\frac{\partial}{\partial x_i} T_f\right)(\phi) = (-1)^1 T_f\left(\frac{\partial}{\partial x_i} \phi\right) = -\int_U f(x) \frac{\partial}{\partial x_i} \phi(x)\, dx$$

$\alpha = (0 \dots \underset{\downarrow i}{1} \dots 0)$

GAUSS GREEN

$$= \int_U \frac{\partial}{\partial x_i} f \; \phi(x)\, dx \qquad (\phi \equiv 0 \text{ on } \partial U)$$

$$\frac{\partial T_f}{\partial x_i} = -T_{\frac{\partial f}{\partial x_i}}$$

Note that $T_m \rightarrow T \Rightarrow D^\alpha T_m \rightarrow D^\alpha T \quad \forall \alpha$

Every distribution T admits derivatives of every order and moreover the order of differentiation does not matter.

In particular since every $f \in L^1_{loc}(\mathbb{R}^n)$ is associated to a distribution T_f then
- f has derivatives in the sense of distributions of every order
  $D^\alpha f$ in the sense of distribution is

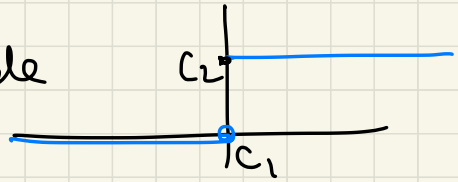
$$D^\alpha T_f(\phi) := (-1)^{|\alpha|} T_f(D^\alpha \phi) = (-1)^{|\alpha|} \int_U f D^\alpha \phi \, dx.$$

Which are the relations with a.e derivatives?

Ex $f = \begin{cases} c_1 & x < x_0 \\ c_2 & x > x_0 \end{cases}$

$c_1, c_2 \in \mathbb{R}$

Heaviside



$$T_f(\phi) = \int_{\mathbb{R}} \phi f(x)\,dx = c_1 \int_{-\infty}^{x_0} \phi(x)\,dx + c_2 \int_{x_0}^{+\infty} \phi(x)\,dx$$

$$T_f'(\phi) = (-1)\, T_f(\phi') = (-1)\cdot c_1 \int_{-\infty}^{x_0} \phi'(x)\,dx + (-1)\, c_2 \int_{x_0}^{+\infty} \phi'(x)\,dx$$

$$= -c_1\left[\phi(x_0) - \cancel{\phi(-\infty)}\right] - c_2\left[\cancel{\phi(+\infty)} - \phi(x_0)\right] = (c_2 - c_1)\,\phi(x_0)$$

The derivative of f in the sense of distribution is $(c_2 - c_1)\delta_{x_0}$ (NOT A FUNCTION!)

↓

note that $f'(x) = 0$ a.e. so the derivative in the sense of distribution is NOT a.e. derivative!

DERIVATIVE IN THE SENSE OF DISTR. IS A DISTRIB. OF ORDER 0

More generally: $f \in \mathcal{C}^1(-\infty, a) \cup (a, +\infty)$ $\quad f^-(a) \neq f^+(a)$

Then $(T_f)'(\varphi) = \int_{\mathbb{R}} \varphi f' \, dx + [f(a^+) - f(a^-)]\, \delta_a(\varphi)$ (exercise)

$= -T_f(\varphi') = -\int_{\mathbb{R}} \varphi' f \, dx \qquad \forall \varphi \in \mathcal{C}^\infty_c(\mathbb{R})$

Ex 2 Let $f(x) = \log|x| \in L^1_{loc}(\mathbb{R})$. Then $f'(x) = \frac{1}{x}$ a.e.

$T_f(\varphi) = \int_{\mathbb{R}} \log|x|\, \varphi(x)\, dx \qquad T'_f(\varphi) = (-1) \cdot \int_{\mathbb{R}} \log|x|\, \varphi'(x)\, dx =$

$= (-1) \lim_{\varepsilon \to 0} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \log|x|\, \varphi'(x)\, dx$

$\int_{\varepsilon}^{+\infty} \log|x|\, \varphi'(x)\, dx = \text{by parts} = \log|x|\, \varphi(x) \Big|_{\varepsilon}^{+\infty} - \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\, dx$

$= \lim_{\varepsilon \to 0} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \frac{\varphi(x)}{x}\, dx + \underbrace{\log(\varepsilon)\varphi(\varepsilon) - \log(\varepsilon)\varphi(-\varepsilon)}_{\to 0}$

$\log(\varepsilon) \cdot \varepsilon \to 0 \qquad \frac{\varphi(\varepsilon) - \varphi(-\varepsilon)}{\varepsilon} \to \text{bdd}$

$= \text{p.v.}\, \frac{1}{x}(\varphi)$

In this case derivative of $f = \lg|x|$ in the sense of distribution coincides with the c.e. derivative (that is with the distribution associated to the c.e. derivative) BUT the c.e. derivative is NOT a $L^1_{loc}(\mathbb{R})$ function.
(DERIVATIVE IN THE SENSE OF DISTRIBUTION IS A DISTR. OF ORDER 1)

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We are interested in analyzing the case in which $f \in L^1_{loc}(\mathbb{R})$ has derivative in the sense of distribution a distribution of 0 order.

Proposition If $f:\mathbb{R}\to\mathbb{R}$ is monotone (non decreasing) then $f\in L^1_{loc}$, f has a countable number of jumps, and $(T_f)'$ is positive, in particular has order 0.

proof JUST of the fact $(T_f')\geq 0$ POSITIVE DISTRIBUTION

Let $\phi\geq 0$ $\phi\in C_c^\infty(\mathbb{R})$. We have to prove that

$T_f'(\phi)\geq 0$ that is $-T_f(\phi')\geq 0 \Rightarrow T_f(\phi')\leq 0$.

$$T_f(\phi')=\int_{\mathbb{R}} f(x)\phi'(x)dx=\lim_{h\to 0^+}\int_{\mathbb{R}} f(x)\frac{\phi(x+h)-\phi(x)}{h}dx=$$

$$=\lim_{h\to 0^+}\frac{1}{h}\left[\int_{\mathbb{R}} f(x)\phi(x+h)dx-\int_{\mathbb{R}} f(x)\phi(x)dx\right]=$$

↓ CHANGE VARIABLE

$$=\lim_{h\to 0^+}\frac{1}{h}\left[\int_{\mathbb{R}} f(x-h)\phi(x)dx-\int_{\mathbb{R}} f(x)\phi(x)dx\right]=$$

$$=\lim_{h\to 0^+}\frac{1}{h}\left[\int_{\mathbb{R}}\left[f(x-h)-f(x)\right]\phi(x)dx\right]\leq 0$$

(since $\phi(x)\geq 0$, $f(x-h)-f(x)\leq 0$)

Moreover $(T_f)'$ positive $\Rightarrow$ $(f_f)'$ of order 0.

(same proof as in the proof of the fact

$T: C_c(\mathbb{R}^n) \to \mathbb{R}$ linear and positive $\Rightarrow \forall K \subset\subset \mathbb{R}^n$

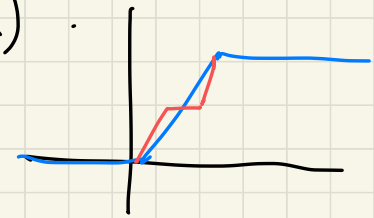
$\exists C_K$ such that $|T(\phi)| \le C_K \|\phi\|_\infty$ $\forall \phi \in C_c(\mathbb{R}^n)$ $\operatorname{supp}\phi \subset K$.

CANTOR VITALI FUNCTION (the DEVIL'S STAIRCASE)

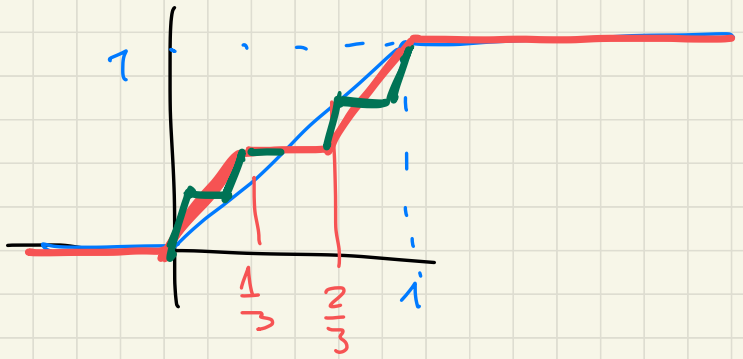
$C = \bigcap C_n$ $\quad C_0 = [0,1]$ $\quad C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$C_n = \frac{1}{3} C_{n-1} \cup \left(\frac{2}{3} + \frac{1}{3}(n-1)\right)$.

$$f_0 = \begin{cases} 0 & x \le 0 \\ x & x \in (0,1) \\ 1 & x \ge 1 \end{cases}$$



$$f_1(x) = \begin{cases} 0 & x \le 0 \\ \frac{3}{2}x & 0 \le x \le \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} \le x \le \frac{2}{3} \\ \frac{3}{2}x - \frac{1}{2} & \frac{2}{3} \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$



$$\begin{cases} f_2 = f_1 \text{ on } C_0 \setminus C_1 \\ f_2 \equiv \left(\frac{1}{2}\right)^2 \text{ on } (C_1 \setminus C_2) \cap \left[0, \frac{1}{3}\right] \\ f_2 = \left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 \text{ on } (C_1 \setminus C_2) \cap \left[\frac{2}{3}, 1\right] \\ \text{then linear} \end{cases}$$

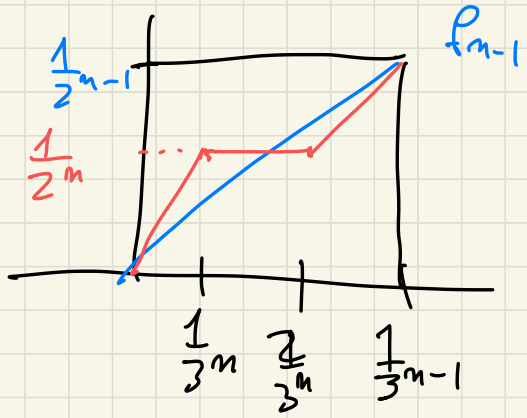
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f_n is continuous, monotone non decreasing and constant on every interval in $\mathbb{R} \setminus C_n$.

$$\|f_n - f_{n-1}\|_\infty \leq \frac{1}{3 \cdot 2^n}$$

$\downarrow$

f_n is a Cauchy sequence in $(C[0,1], \|\cdot\|_\infty)$



$f_n \to f$ uniformly, $f : \mathbb{R} \to [0,1]$

$f(x) \equiv 0 \quad x \leq 0$

$f(x) \equiv 1 \quad x \geq 1$

f is monotone non decreasing

Continuous

and constant on every interval in $\mathbb{R} \setminus C$

(C = Cantor set).