COMPUTABILITY (02/11/2026)

* Diagonalisation

idea: $x_i \quad i \in I$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad \dots \quad x_i \quad \dots$

$\}$ position k of x_i

construct x s.t.

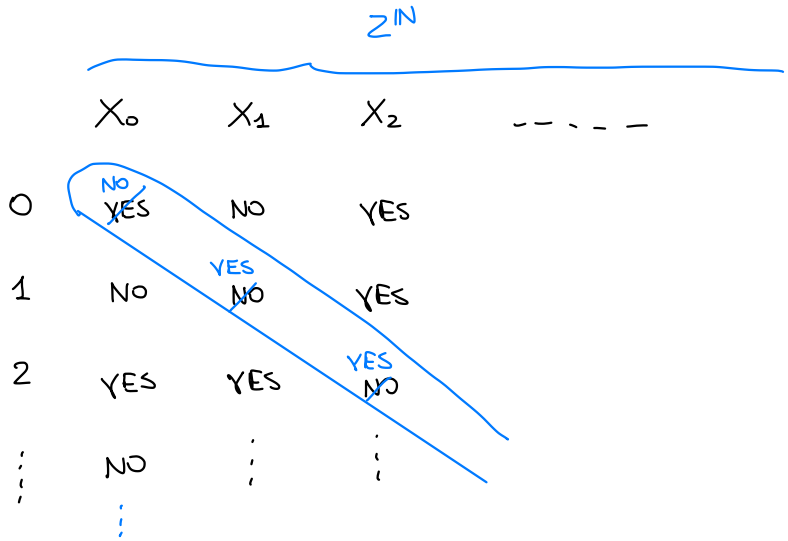
$x \neq x_i \quad \forall i \in I$ — x differs from x_i at position i

[Cantor (XIX - XX century, modern set theory, infinite cardinalities)]

OBSERVATION: $\boxed{|\mathbb{N}| < |2^{\mathbb{N}}|}$ $\qquad 2^X = \{Y \mid Y \subseteq X\}$

proof

assume $|\mathbb{N}| \geq |2^{\mathbb{N}}|$ i.e. there is a surjective function $\mathbb{N} \to 2^{\mathbb{N}}$



$X_0 = \{0, 2\}$

$D = \{ i \mid i \notin X_i \} \subseteq \mathbb{N}$

$\rightsquigarrow \exists k \in \mathbb{N}$ s.t. $D = X_k$

does k belong to D ? $\quad k \in D$?

- YES : $\quad k \in D \Rightarrow k \notin X_k = D$ $\quad$ contradiction
- NO : $\quad k \notin D \Rightarrow k \in X_k = D$ $\quad$ "

$\rightsquigarrow |2^{\mathbb{N}}| > |\mathbb{N}|$, $2^{\mathbb{N}}$ is not countable $\square$

EXERCISE : $\mathcal{Y} = \{ f \mid f : \mathbb{N} \to \mathbb{N} \}$ is not countable, i.e.

$|\mathcal{Y}| > |\mathbb{N}|$

(1st proof)

$$\mathcal{Y}_2 = \left\{ f \in \mathcal{Y} \mid \begin{array}{l} f : \mathbb{N} \to \mathbb{N} \text{ total} \\ \mathrm{cod}(f) = \{0,1\} \end{array} \right\} \subseteq \mathcal{Y}$$

$\mathcal{Y}_2 \subseteq \mathcal{Y}$

$\rightsquigarrow$ $\quad |\mathcal{Y}_2| = |2^{\mathbb{N}}|$ $\quad$ bijection $\quad \mathcal{Y}_2 \to 2^{\mathbb{N}}$

$\qquad\qquad\quad \wedge \qquad\quad \vee \qquad\qquad f \mapsto \{ i \mid f(i) = 1 \}$

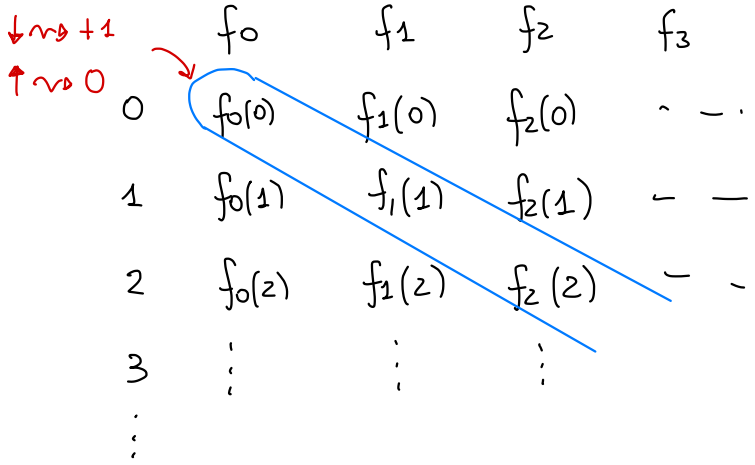
$\mathcal{Y}_2 \to \mathcal{Y}$ injective $f \mapsto f$ $\quad |\mathcal{Y}| \qquad |\mathbb{N}|$

$\qquad\qquad\quad \Downarrow$

$|\mathcal{Y}| > |\mathbb{N}|$

(2nd proof) $\quad |\mathcal{Y}| > |\mathbb{N}|$

consider an enumeration of functions in $\mathcal{Y}$



define $\quad d : \mathbb{N} \to \mathbb{N}$

$$d(x) = \begin{cases} f_x(x) + 1 & \text{if } f_x(x) \downarrow \\ 0 & \text{if } f_x(x) \uparrow \end{cases}$$

then $d \neq f_x \quad \forall x \in \mathbb{N}$

since d is total

if f_x is total then $d \neq f_x$ because $d(x) = f_x(x) + 1 \neq f_x(x)$

Hence there is no full enumeration of $\mathcal{G}$, i.e. $|\mathcal{G}| > |\mathbb{N}|$

□

OBSERVATION: There is a total non computable function

$$f : \mathbb{N} \to \mathbb{N}$$

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_x(x) \downarrow \quad (x \in W_x) \\ 0 & \text{if } \varphi_x(x) \uparrow \quad (x \notin W_x) \end{cases}$$

	φ_0	φ_1	φ_2	φ_3	- - -
0	$\varphi_0(0)$	$\varphi_1(0)$	$\varphi_2(0)$		- -
1	$\varphi_0(1)$	$\varphi_1(1)$	$\varphi_2(1)$		- - -
2	$\varphi_0(2)$	$\varphi_1(2)$	$\varphi_2(2)$		- -
3	⋮	⋮	⋮		

- f is total
- $\forall x \in \mathbb{N}$ if φ_x is total then $\varphi_x \neq f$ since $\varphi_x(x) \neq f(x)$

  in fact, since φ_x is total $\varphi_x(x) \downarrow$

  and thus $f(x) = \varphi_x(x) + 1 \neq \varphi_x(x)$

↳ f total, different from all total computable functions

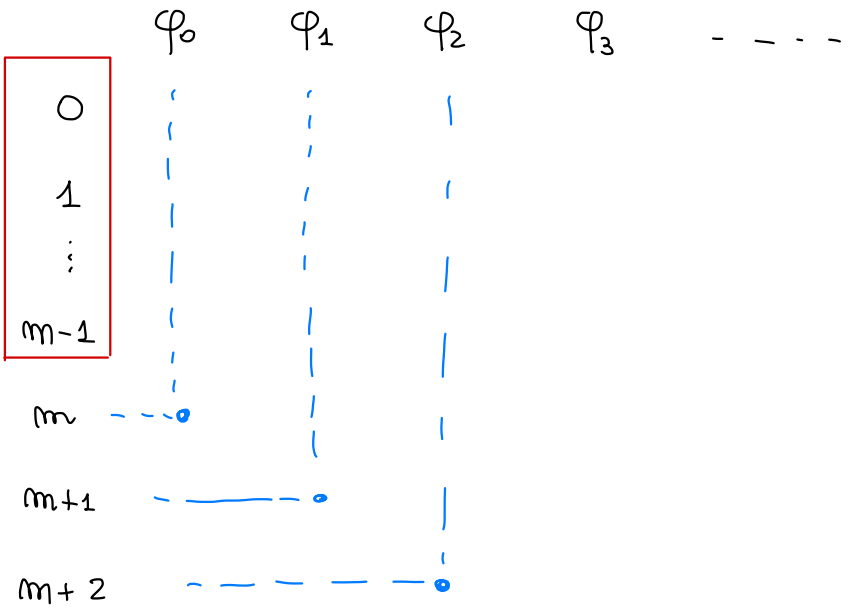
⇒ f not computable.

□

EXERCISE: let $m \in \mathbb{N}$, $f : \mathbb{N} \to \mathbb{N}$ given function

show that there is $g : \mathbb{N} \to \mathbb{N}$ not computable such that

$$g(x) = f(x) \qquad \forall x < m$$



$$g(x) = \begin{cases} f(x) & \text{if } x < m \\ \varphi_{x-m}(x) + 1 & \text{if } x \geq m \text{ and } \varphi_{x-m}(x)\downarrow \\ 0 & \text{if } x \geq m \text{ and } \varphi_{x-m}(x)\uparrow \end{cases}$$

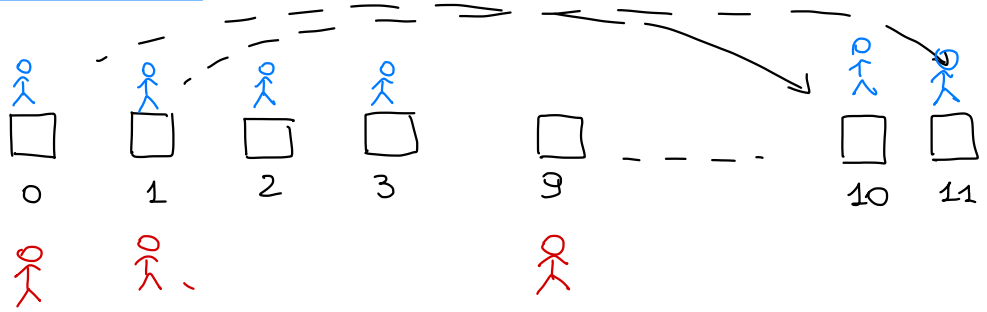
hence $g(x) = f(x) \qquad x < m$

$\forall x \in \mathbb{N} \quad g \neq \varphi_x$ since $g(x+m) \neq \varphi_x(x+m)$

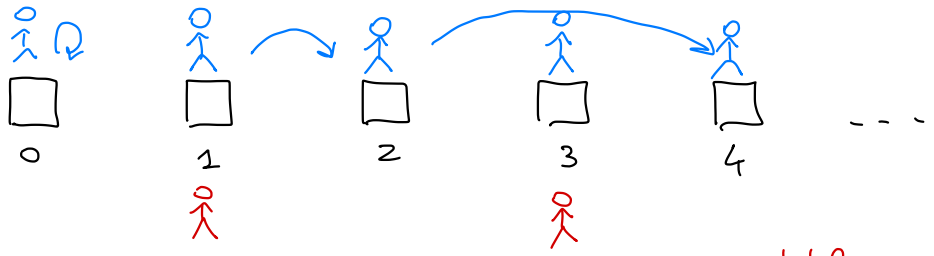
in fact if $\varphi_x(x+m)\downarrow$ then $g(x+m) = \varphi_x(x+m) + 1 \neq \varphi_x(x+m)$

if $\varphi_x(x+m)\uparrow$ then $g(x+m) = 0 \neq \varphi_x(x+m)$

Hilbert Hotel



$m = 10$ new guests



countably many new guests

Alternative

$$g(x) = \begin{cases} f(x) & \text{if } x < m \\ \varphi_x(x) + 1 & \text{if } x \geq m \text{ and } \varphi_x(x)\downarrow \\ 0 & \text{if } x \geq m \text{ and } \varphi_x(x)\uparrow \end{cases}$$

g is not computable

$g \neq \varphi_x \quad x \geq m$

$\varphi_0 \quad \varphi_1 \quad \varphi_2 \quad \cdots \quad \varphi_{m-1} \quad \varphi_m \quad \varphi_{m+1} \quad \varphi_{m+2} \quad \cdots$

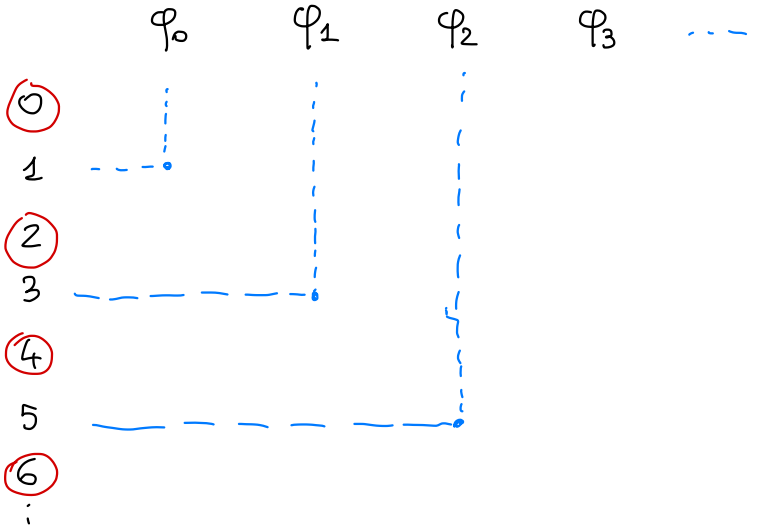
↑ there are infinitely many occurrences of each computable function

for all computable functions h $\exists x \geq m$ s.t. $h = \varphi_x$

$\Rightarrow \quad g \neq \varphi_x = h \quad \Rightarrow \quad g \neq h$

$\Rightarrow$ g different from all computable functions is not computable!

EXERCISE: show that there is a function $g: \mathbb{N} \to \mathbb{N}$ total not computable and such that $g(x) = 0 \quad \forall x \in \mathbb{N}$ x even



$$g(x) = \begin{cases} 0 & \text{if } x \text{ even} \\ \varphi_{\frac{x-1}{2}}(x) + 1 & \text{if } x \text{ odd and } \varphi_{\frac{x-1}{2}}(x) \downarrow \\ 0 & \text{if } x \text{ odd and } \varphi_{\frac{x-1}{2}}(x) \uparrow \end{cases}$$

→ g total

→ $g(x) = 0 \quad \forall x \in \mathbb{N}$ x even

→ g is not computable since $g \neq \varphi_x \quad \forall x \in \mathbb{N}$

$g(2x+1) \neq \varphi_x(2x+1)$

NOTE: In this case

$$g(x) = \begin{cases} 0 & \text{if } x \text{ even} \\ \varphi_x(x) + 1 & \text{if } x \text{ odd and } \varphi_x(x) \downarrow \\ 0 & \text{if } x \text{ odd and } \varphi_x(x) \uparrow \end{cases}$$

↳ unclear whether this works (i.e. g not computable)

EXERCISE: Given $f_0, f_1, f_2, \ldots : \mathbb{N} \to \mathbb{N}$ $(f_i)_{i \in \mathbb{N}}$

Define $f : \mathbb{N} \to \mathbb{N}$ s.t. $\text{dom}(f) \neq \text{dom}(f_i) \quad \forall i \in \mathbb{N}$

PARAMETRISATION (SMN) THEOREM

Let $e \in \mathbb{N}$ be a program code

and consider the function of two arguments computed by $P_e = \gamma^{-1}(e)$

$$\varphi_e^{(2)} : \mathbb{N}^2 \to \mathbb{N}$$

$$f(x,y) = \varphi_e^{(2)}(x,y)$$

Let $x \in \mathbb{N}$ be fixed

$$f_x : \mathbb{N} \to \mathbb{N}$$

$$f_x(y) = f(x,y) = \varphi_e^{(2)}(x,y) \qquad \text{computable}$$

e.g. $f(x,y) = y^x$

$$f_0(y) = y^0 = 1$$

$$f_1(y) = y^1 = y$$

$$f_2(y) = y^2$$

$$\vdots$$

since f_x is computable for all $x \in \mathbb{N}$ fixed there is $d \in \mathbb{N}$

$$f_x = \varphi_d = \varphi_{s(e,x)}$$

this depends on e, x

hence there is a total function $s : \mathbb{N}^2 \to \mathbb{N}$

$$s(e,x) = d$$

The smn theorem says that $s : \mathbb{N}^2 \to \mathbb{N}$ is computable

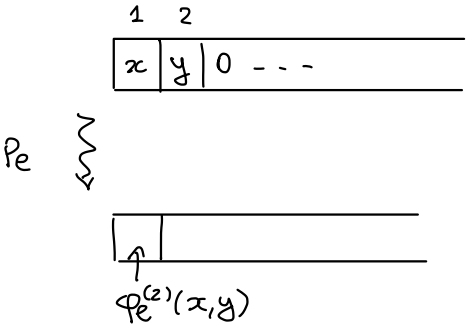
```
def Pe(x,y):
    ⋮
    x
    y
    return ...
```

$x = 2$

```
def Pe(x,y):
    ⋮
    2
    y
    return ...
```

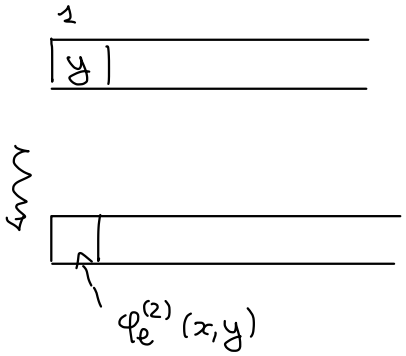
Idea :

given $e \in \mathbb{N}$ (P_e)



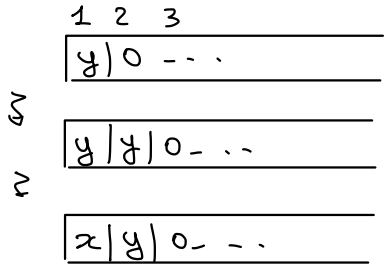
now given $e \in \mathbb{N}$ and $x \in \mathbb{N}$

we want to determine $P_{S(e,x)}$



how can you do that

- → move y to R_2
- → write x in register R_1 (*)
- → execute $P_e = \gamma^{-1}(e)$



$$S(e,x) = \gamma(*)$$

[formal proof, next lesson]